1. (3.5 pts.) In this exercise let us consider a CSP with variables \( x, y, u, v \), domains \( \{1, 3\} \), \( \{1, 2, 3, 4, 5, 6\} \), \( \{1, 4\} \) and \( \{1, 3, 4\} \), respectively, and a single constraint \text{alldifferent}(x, y, u, v).

(a) (0.5 pts.) Let \( G \) be the value graph of the \text{alldifferent} constraint. Draw \( G \) and give the list of its edges.

(b) (0.5 pts.) Let us consider the assignment \((x, y, u, v) = (3, 6, 1, 4)\), which is a solution to the CSP. Let \( M \) be the matching covering \( x, y, u, v \) that corresponds to this solution. Draw \( M \). Only draw the vertices of \( G \) and the edges of the matching.

(c) (0.5 pts.) Let \( G' \) be the directed graph in which the edges of \( G \) have been oriented according to matching \( M \). Draw \( G' \).

(d) (0.5 pts.) Which are the strongly connected components of \( G' \)? Draw the vertices of \( G' \) and those edges with source and destination in the same strongly connected component.

(e) (0.5 pts.) Which are the free vertices with respect to \( M \)? Find the simple paths in \( G' \) that start at these free vertices.

(f) (0.5 pts.) Are there any vital edges? If so, which? Justify the answer.

(g) (0.5 pts.) Which values can be removed from the domains of which variables by applying arc consistency to the \text{alldifferent} constraint? Justify the answer.

2. (3.5 pts.) An electronic circuit is modeled with a directed acyclic graph \( G = (V, E) \), where \( V \) are the components of the circuit (resistors, transistors, capacitors, etc.), and \( E \) are the wires that connect the components. Namely, a wire \( e = (u, v) \in E \) sends the current signal produced as output at component \( u \) as an input signal to component \( v \). The amount of time this transmission takes is called the delay of \( e \), which we denote by \( c_e \). For the output signal at a component to be produced, the component needs to wait until there is current in all of its incoming wires.

There are two distinguished components: the input component \( s \), which is the only component without incoming wires (let us assume that it generates the input current signal from the environment); and the output component \( t \), which is the only component without outgoing wires (let us assume that it transparently transmits the output current signal back to the environment).
We are interested in, given a circuit and the delays of the wires, to find out the least amount of time that it can take for the current signal to go from the input component to the output component.

(a) (1 pt.) Consider real variables $y_u$ for every $u \in V$ meaning “the time at which the current signal leaves component $u$”. Model the problem as a (continuous) linear program. If there are sign constraints, make them explicit.

(b) (1 pt.) Let us refer to the problem in the previous exercise as $P$. Let $D$ be its dual problem. Formulate $D$: indicate the dual variables, the dual objective function and the dual constraints. If there are sign constraints, make them explicit.

(c) (1 pt.) Prove that feasible dual solutions (i.e., solutions to the dual problem) are integer.

(d) (0.5 pts.) Is there any computational advantage as regards efficiency in formulating and solving $D$ as opposed to simply applying the primal simplex algorithm to $P$? Which algorithm would you use to exploit this computational advantage? Justify the answer.

3. (3 pts.) In SAT solving, vivification is the following process used for simplifying the formula. Let $F$ be a formula in CNF which is given as input to the SAT solver, and let $C = \ell_1 \lor \ell_2 \lor \ldots \lor \ell_k \in F$ be a clause. Given a formula $G$, let us denote by $\text{UP}(G)$ the set of literals that can be obtained by applying unit propagation to $G$ exhaustively. Clause $C$ is vivified if for some $i \in \{1, \ldots, k-1\}$, the set $S_i = \text{UP}(F - \{C\} \cup \{-\ell_1, -\ell_2, \ldots, -\ell_i\})$ satisfies one of the following conditions:

(A) For some $j > i$, $\ell_j \in S_i$.

(B) For some $j > i$, $-\ell_j \in S_i$.

(C) For some literal $\ell$, we have $\ell, -\ell \in S_i$.

(a) (1 pt.) In case (A), is it possible to replace in $F$ clause $C$ by another simpler clause $C'$ while preserving the logical equivalence of the formula? Which clause would $C'$ be? Justify your answer.

(b) (1 pt.) Idem for case (B).

(c) (1 pt.) Idem for case (C).