1. (3 pts.) Let $x$ and $y$ be integer variables with domains $[L_x, U_x]$ and $[L_y, U_y]$, respectively (where $L_x, U_x, L_y, U_y \in \mathbb{Z}$, $L_x \leq U_x$ and $L_y \leq U_y$). Let us consider the constraint $ax - by = c$, where $a, b > 0$ and $a, b, c \in \mathbb{Z}$.

(a) (1 pt.) Prove that a value $Y$ for $y$ is arc-consistent with respect to $x$ if and only if $aL_x - bY \leq c \leq aU_x - bY$ and $a \mid (c + bY)$ (where $\mid$ means “divides”).

(b) (1 pt.) Prove that variable $y$ is bounds-consistent with respect to $x$ if and only if $aL_x - bL_y \leq c \leq aU_x - bU_y$ and $a \mid (c + bL_y)$ and $a \mid (c + bU_y)$.

(c) (1 pt.) Prove that the constraint $ax - by = c$ is bounds-consistent if and only if $aL_x - bL_y = c = aU_x - bU_y$ and $a \mid (c + bL_y)$ and $a \mid (c + bU_y)$ and $b \mid (-c + aL_x)$ and $b \mid (-c + aU_x)$.

2. (2.1 pts.) Let us consider the following linear program:

\[
\begin{align*}
\text{min} \quad & -2x_0 - x_3 \\
\text{such that} \quad & -2x_0 - x_1 - x_2 + 2x_3 = -2 \\
& x_0 \geq 0, \; x_0 \in \mathbb{R} \\
& x_1 \geq 0, \; x_1 \in \mathbb{R} \\
& x_2 \geq 0, \; x_2 \in \mathbb{R} \\
& x_3 \geq 0, \; x_3 \in \mathbb{R}
\end{align*}
\]

Variables $(x_3, x_2)$ form a feasible basis for the simplex algorithm (you do not need to prove that).

(a) The basic solution assigns

- (0.3 pts.) basic variable $x_3$ to value \\
- (0.3 pts.) basic variable $x_2$ to value \\

(b) (0.3 pts.) The value of the objective function at the basic solution is \\

(c) The reduced cost of

- (0.3 pts.) non-basic variable $x_0$ is \\
- (0.3 pts.) non-basic variable $x_1$ is \\

(d) (0.3 pts.) Is non-basic variable $x_0$ satisfying the optimality conditions? (answer yes/no)
If it does not, give the best value it can be assigned to according to the ratio test
(answer $+\infty$ if no such value exists).

(e) (0.3 pts.) Is non-basic variable $x_1$ satisfying the optimality conditions? (answer yes/no)
If it does not, give the best value it can be assigned to according to the ratio test
(answer $+\infty$ if no such value exists).

3. (1.9 pts.) Let us consider the following linear program:

$$\begin{align*}
\text{min} & \quad -2x_0 - 2x_1 + 2x_2 + 2x_3 \\
\text{subject to} & \quad x_0 + x_1 + 2x_3 = 0 \\
& \quad x_0 - 2x_1 + x_3 = 2 \\
& \quad -x_1 + x_2 + 2x_3 = -2 \\
& \quad -6 \leq x_0 \leq -4, \quad x_0 \in \mathbb{R} \\
& \quad -2 \leq x_1 \leq -1, \quad x_1 \in \mathbb{R} \\
& \quad -14 \leq x_2 \leq -11, \quad x_2 \in \mathbb{R} \\
& \quad 4 \leq x_3 \leq 5, \quad x_3 \in \mathbb{R}
\end{align*}$$

Variables $(x_2, x_0, x_3)$ form a basis for the bounded simplex algorithm. If

- non-basic variable $x_1$ is assigned to value $-2$

the basic solution is feasible (you do not need to prove that).

(a) The basic solution assigns

- (0.3 pts.) basic variable $x_2$ to value
- (0.3 pts.) basic variable $x_0$ to value
- (0.3 pts.) basic variable $x_3$ to value

(b) (0.3 pts.) The value of the objective function at the basic solution is

(c) The reduced cost of

- (0.3 pts.) non-basic variable $x_1$ is

(d) (0.4 pts.) Is non-basic variable $x_1$ satisfying the optimality conditions? (answer yes/no)
If it does not, give the best value it can be assigned to according to the ratio test.

4. (3 pts.) We say a literal in a formula in CNF is pure if it appears always positively or always negatively.

(a) (2 pts.) Show that, if $C'$ is a clause that contains at least one pure literal in the CNF
$\{C_1, \ldots, C_n, C'\}$, then this CNF is satisfiable if and only if $\{C_1, \ldots, C_n\}$ is satisfiable.

(b) (1 pt.) Explain how pure literals can be exploited in a preprocessing step of the formula in
a SAT solver. Explain also if this affects in any way the generation of models by the SAT solver.