1. (0.9 pts.) Let us define a binary CSP $C$ with:
   - variable $x_0$ with domain $\{0, 1, 3\}$
   - variable $x_2$ with domain $\{0, 1, 2\}$
   - variable $x_4$ with domain $\{0, 2, 3\}$
   - constraint $x_0 \leq x_2$
   - constraint $x_4 \neq x_0$
   - constraint $x_4 = x_2$

Write the domains of the variables after enforcing arc-consistency on $C$:

(a) (0.3 pts.) variable $x_0$ has domain

(b) (0.3 pts.) variable $x_2$ has domain

(c) (0.3 pts.) variable $x_4$ has domain

Use the algorithm for enforcing arc-consistency that you prefer.

Solution:

(a) variable $x_0$ has domain $\{0, 1\}$
(b) variable $x_2$ has domain $\{0, 2\}$
(c) variable $x_4$ has domain $\{0, 2\}$

2. (0.9 pts.) Let us consider the same binary CSP $C$ as in exercise 1.

Write the domains of the variables after enforcing directional arc-consistency on $C$ with the ordering $x_0 \prec x_2 \prec x_4$:

(a) (0.3 pts.) variable $x_0$ has domain

(b) (0.3 pts.) variable $x_2$ has domain

(c) (0.3 pts.) variable $x_4$ has domain
Use the algorithm for enforcing directional arc-consistency that you prefer.

**Solution:**

(a) variable \( x_0 \) has domain \{0, 1\}
(b) variable \( x_2 \) has domain \{0, 2\}
(c) variable \( x_4 \) has domain \{0, 2, 3\}

3. (0.9 pts.) Let us consider the same binary CSP \( C \) as in exercise 1.

Write the domains of the variables after enforcing **bounded arc-consistency** on \( C \):

(a) (0.3 pts.) variable \( x_0 \) has domain ______________________
(b) (0.3 pts.) variable \( x_2 \) has domain ______________________
(c) (0.3 pts.) variable \( x_4 \) has domain ______________________

Use the algorithm for enforcing bounded arc-consistency that you prefer.

**Solution:**

(a) variable \( x_0 \) has domain \{0, 1\}
(b) variable \( x_2 \) has domain \{0, 1, 2\}
(c) variable \( x_4 \) has domain \{0, 2\}

4. (0.9 pts.) Let us consider the same binary CSP \( C \) as in exercise 1.

Write the domains of the variables after enforcing **singleton arc-consistency** on \( C \):

(a) (0.3 pts.) variable \( x_0 \) has domain ______________________
(b) (0.3 pts.) variable \( x_2 \) has domain ______________________
(c) (0.3 pts.) variable \( x_4 \) has domain ______________________

Use the algorithm for enforcing singleton arc-consistency that you prefer.

**Solution:**

(a) variable \( x_0 \) has domain \{0, 1\}
(b) variable \( x_2 \) has domain \{2\}
(c) variable \( x_4 \) has domain \{2\}
5. (2.1 pts.) Let us consider the following linear program:

\[
\begin{align*}
\text{min} & \quad x_0 + 2x_2 - x_3 \\
\text{such that} & \\
-2x_0 - x_1 + x_2 + x_3 &= 0 \\
x_1 - 2x_2 &= -2 \\
x_0 &\geq 0, \quad x_0 \in \mathbb{R} \\
x_1 &\geq 0, \quad x_1 \in \mathbb{R} \\
x_2 &\geq 0, \quad x_2 \in \mathbb{R} \\
x_3 &\geq 0, \quad x_3 \in \mathbb{R}
\end{align*}
\]

Variables \((x_2, x_1)\) form a feasible basis for the simplex algorithm (you do not need to prove that).

(a) The basic solution assigns
- (0.3 pts.) basic variable \(x_2\) to value \(2\)
- (0.3 pts.) basic variable \(x_1\) to value \(2\)

(b) (0.3 pts.) The value of the objective function at the basic solution is \(4\)

(c) The reduced cost of
- (0.3 pts.) non-basic variable \(x_0\) is \(-3\)
- (0.3 pts.) non-basic variable \(x_3\) is \(1\)

(d) (0.3 pts.) Is non-basic variable \(x_0\) satisfying the optimality conditions? (answer \textbf{yes}/\textbf{no})
If it does not, give the best value it can be assigned to according to the ratio test (answer \(+\infty\) if no such value exists).

\(-\)

(e) (0.3 pts.) Is non-basic variable \(x_3\) satisfying the optimality conditions? (answer \textbf{yes}/\textbf{no})
If it does not, give the best value it can be assigned to according to the ratio test (answer \(+\infty\) if no such value exists).

\(-\)

Solution:

(a) The basic solution assigns
- basic variable \(x_2\) to value 2
- basic variable \(x_1\) to value 2

(b) The value of the objective function at the basic solution is 4

(c) The reduced cost of
- non-basic variable \(x_0\) is \(-3\)
- non-basic variable \(x_3\) is 1

(d) Non-basic variable \(x_0\) does not satisfy the optimality conditions and the best value it can be assigned to is \(1/2\) .
(e) Non-basic variable $x_3$ satisfies the optimality conditions.

6. (1.8 pts.) Let us consider the following linear program:

$$
\begin{align*}
\text{min} & \quad x_1 - 2x_3 \\
\text{such that} & \quad -x_1 + 2x_2 + x_3 = -2 \\
& \quad x_0 - 2x_1 - 2x_2 - x_3 = 1 \\
& \quad -x_0 - 2x_1 + x_2 + x_3 = 1 \\
& \quad 0 \leq x_0 \leq 2, \quad x_0 \in \mathbb{R} \\
& \quad 1 \leq x_1 \leq 2, \quad x_1 \in \mathbb{R} \\
& \quad -6 \leq x_2 \leq -5, \quad x_2 \in \mathbb{R} \\
& \quad 11 \leq x_3 \leq 12, \quad x_3 \in \mathbb{R}
\end{align*}
$$

Variables $(x_3, x_2, x_0)$ form a basis for the bounded simplex algorithm. If

- non-basic variable $x_1$ is assigned to value 1

the basic solution is feasible (you do not need to prove that).

(a) The basic solution assigns
- (0.3 pts.) basic variable $x_3$ to value $11$
- (0.3 pts.) basic variable $x_2$ to value $-6$
- (0.3 pts.) basic variable $x_0$ to value $2$

(b) (0.3 pts.) The value of the objective function at the basic solution is $-21$

(c) The reduced cost of
- (0.3 pts.) non-basic variable $x_1$ is $-17$

(d) (0.3 pts.) Is non-basic variable $x_1$ satisfying the optimality conditions? (answer yes/no) If it does not, give the best value it can be assigned to according to the ratio test.

Solution:

(a) The basic solution assigns
- basic variable $x_3$ to value 11
- basic variable $x_2$ to value $-6$
- basic variable $x_0$ to value 2

(b) The value of the objective function at the basic solution is $-21$

(c) The reduced cost of
- non-basic variable $x_1$ is $-17$

(d) Non-basic variable $x_1$ does not satisfy the optimality conditions and the best value it can be assigned to is 1.
7. (2.5 pts.) Let $C$ be a constraint defined over Boolean variables $x_1, x_2, \ldots, x_n$, and let $S$ be a CNF defined over $x_1, x_2, \ldots, x_n$ and possibly also over additional Boolean variables $y_1, y_2, \ldots, y_m$. We say that $S$ is a correct encoding for $C$ if:

(i) any assignment over $x_1, x_2, \ldots, x_n$ that satisfies $C$ can be extended to an assignment over $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ that satisfies $S$; and

(ii) any assignment over $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ that satisfies $S$ also satisfies $C$.

In this exercise we will focus on a particular kind of constraint: given $n$ Boolean variables $x_1, x_2, \ldots, x_n$ (where $n \geq 3$), we define $AMT(x_1, x_2, \ldots, x_n)$ (At Most Two) as the constraint

$$x_1 + x_2 + \ldots + x_n \leq 2$$

(a) (0.5 pts.) Let us consider the CNF consisting of all clauses

$$x_i \lor x_j \lor x_k$$

where $1 \leq i < j < k \leq n$. It can be shown (but you do not have to prove it) that this is a correct encoding for $AMT(x_1, x_2, \ldots, x_n)$, which we will refer to as the naive encoding.

What is the total number of clauses in terms of $n$? $\Theta(\quad )$.

(b) (1 pt.) Let us introduce the small-step encoding for $AMT(x_1, x_2, \ldots, x_n)$. The CNF $S$ is defined recursively as follows:

- If $n = 3$ then $S$ consists of the clauses of the naive encoding for $AMT(x_1, x_2, \ldots, x_n)$.
- If $n > 3$ then $S$ consists of:
  - the clauses of $AMT(x_1, x_2, \ldots, x_{n-1})$ using recursively the small-step encoding, and
  - a clause $\overline{x_n} \lor C$ for each clause $C$ in Heule’s encoding for $AMO(x_1, x_2, \ldots, x_{n-1})$.

What is the total number of additional variables in terms of $n$? $\Theta(\quad )$

What is the total number of clauses in terms of $n$? $\Theta(\quad )$.

(c) (1 pt.) Prove that the small-step encoding for $AMT(x_1, x_2, \ldots, x_n)$ is correct. You can use that Heule’s encoding is correct without having to prove it.
Solution:

(a) The total number of clauses of the naive encoding in terms of $n$ is \( \binom{n}{3} = \Theta(n^3) \).

(b) After unfolding the recursion, the encoding consists of:

(i) clauses of the naive encoding of $AMT(x_1, x_2, x_3)$, that is, the clause $\neg x_1 \lor \neg x_2 \lor \neg x_3$.

(ii) clauses of the form $\overline{x_k} \lor C$, where $4 \leq k \leq n$, and $C$ is a clause from Heule’s encoding of $AMO(x_1, x_2, \ldots, x_{k-1})$.

There is a single clause of type (i), which does not introduce additional variables. As regards clauses of type (ii), we have as many clauses as clauses in the union of Heule’s encodings of $AMO(x_1, x_2, \ldots, x_{k-1})$ for $4 \leq k \leq n$. Since Heule’s encoding of an $AMO$ over $k$ variables uses $\Theta(k)$ clauses and additional variables, in total we have $\Theta(3 + 4 + \ldots + n - 1) = \Theta(n^2)$ clauses and additional variables.

(c) Let $I$ be a model of the small-step encoding. Let us assume that $I(x_1) + I(x_2) + \ldots + I(x_n) > 2$ and we will get a contradiction. Let $i, j, k$ with $1 \leq i < j < k \leq n$ be such that $I(x_i) = I(x_j) = I(x_k) = 1$. Since the small-step encoding contains clause $\neg x_1 \lor \neg x_2 \lor \neg x_3$, it must be $k \geq 4$. But the encoding also includes clauses of the form $\overline{x_k} \lor C$, where $C$ is a clause in Heule’s encoding of $AMO(x_1, x_2, \ldots, x_{k-1})$. Since $I$ is a model of the encoding, $I$...
satisfies each of these clauses. And as $I(x_k) = 1$, we have that $I$ satisfies $C$ for any clause $C$ in Heule’s encoding of $AMO(x_1, x_2, \ldots, x_{k-1})$. This is impossible, as Heule’s encoding is correct, $I(x_i) = I(x_j) = 1$ and $1 \leq i < j \leq k - 1$.

Now let us assume that we have an assignment $I$ on $x_1, x_2, \ldots, x_n$ such that $I(x_1) + I(x_2) + \ldots + I(x_n) \leq 2$. Then there are $i, j$ with $1 \leq i < j \leq n$ such that for any $1 \leq k \leq n$, if $k \neq i$ and $k \neq j$ then $I(x_k) = 0$. We have that the clause $\overline{x_i} \lor \overline{x_j} \lor x_k$ of the base case of the small-step encoding is satisfied no matter how the additional variables are assigned.

Now let us consider a clause of the form $\overline{x_k} \lor C$, where $4 \leq k \leq n$ and $C$ belongs to Heule’s encoding of $AMO(x_1, x_2, \ldots, x_{k-1})$. If $k \neq i$ and $k \neq j$ then $I(x_k) = 0$ and the clause is satisfied. If $k = i$, since $I(x_p) = 0$ for any $1 \leq p \leq i - 1$ and Heule’s encoding is correct, we have that we can assign values to the additional variables so that all clauses of $AMO(x_1, x_2, \ldots, x_{k-1})$ are satisfied. Finally if $k = j$, since $I(x_p) = 0$ for any $1 \leq p \leq j - 1$ and $p \neq i$, and as Heule’s encoding is correct, we have that we can assign values to the additional variables so that all clauses of $AMO(x_1, x_2, \ldots, x_{k-1})$ are satisfied. Altogether, $I$ can be extended to the additional variables so that any clause $C$ belonging to Heule’s encoding of $AMO(x_1, x_2, \ldots, x_{k-1})$, where $4 \leq k \leq n$, is satisfied.