1. (3 pts.) Let $x_1, \ldots, x_n$ be integer variables. For each $1 \leq i \leq n$, the domain of variable $x_i$ is the interval $[\ell_i, u_i]$, where $\ell_i, u_i \in \mathbb{Z}$ and $\ell_i \leq u_i$.

Let $C$ be a constraint of the form $a_1 x_1 + \cdots + a_n x_n \geq k$, where $a_i, k \in \mathbb{Z}$ and $a_i > 0$.

Let us consider the CSP consisting of the single constraint $C$.

(a) (1 pt.) Prove that, if $x$ is a solution to the CSP, then for each $1 \leq i \leq n$ we have that

\[ x_i \geq \left\lceil \frac{k - \sum_{j=1, j \neq i}^{n} a_j u_j}{a_i} \right\rceil \]

(b) (1 pt.) Let $S = \sum_{j=1}^{n} a_j u_j$. Prove that, if $S - \max_{1 \leq j \leq n} (a_j (u_j - l_j)) < k$, then the CSP is arc-inconsistent.

(c) (1 pt.) Is the reverse implication of exercise (1b) true? If so, prove it. Otherwise, give a counterexample.

2. (3 pts.) Consider an integer linear program of the following form:

\[
\begin{align*}
& \min c^T x \\
& Ax = b \\
& x \geq 0 \quad x_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n.
\end{align*}
\]

Let $\beta$ be a basic solution and

\[ x_i = \gamma - \sum_{j \in R} a_j x_j \]

be the equation in the tableau of a basic variable $x_i$, where $R$ are the indices of the non-basic variables and $\gamma, a_j \in \mathbb{R}$. Let us assume that $\beta_i \notin \mathbb{Z}$, where $\beta_i$ is the value assigned by $\beta$ to $x_i$.

(a) (1.5 pts.) Prove that

\[ x_i + \sum_{j \in R} [a_j] x_j - \lfloor \gamma \rfloor = \gamma - \lfloor \gamma \rfloor - \sum_{j \in R} (a_j - \lfloor a_j \rfloor) x_j \]

for all feasible solutions to the integer linear program.

(b) (1.5 pts.) Prove that

\[ \gamma - \lfloor \gamma \rfloor - \sum_{j \in R} (a_j - \lfloor a_j \rfloor) x_j \leq 0 \]

is a cut that cuts $\beta$ away.
3. (4 pts.) Answer the following questions:

(a) (1.5 pts.) Using the standard representation of a 2-comparator with inputs $x_1, x_2$ and outputs $y_1, y_2$ (in decreasing order):

```
  x1 -- y1
  x2 -- y2
```

draw the circuit corresponding to a sorting network with 4 inputs $v_1, v_2, v_3, v_4$. Please indicate clearly the names of the auxiliary variables representing the wires of the circuit. Write also the set of clauses corresponding to the circuit.

Note: You can write the clauses as disjunctions or as implications.

(b) (1.5 pts.) A pseudo-boolean constraint is a constraint of the form $a_1 x_1 + \cdots + a_m x_m \leq k$, where $k, a_1, \ldots, a_m$ are positive integers and $x_1, \ldots, x_m$ are boolean variables. Explain how to encode into SAT a pseudo-boolean constraint using sorting networks. Illustrate your method with the constraint $v_1 + 3v_2 \leq 2$ and give the resulting set of clauses.

(c) (1 pt.) Is your encoding of exercise (3b) arc-consistent? (that is, if a value of a variable does not have a support for the constraint, does unit propagation in the CNF propagate a literal that discards that value?)

If it is so, prove it. Otherwise, give a counterexample.