1. (3 points) Let \( x, y \) be variables with finite domains \( D_x, D_y \subseteq \mathbb{Z} \).

   (a) (1.5 points) Let us consider the CSP consisting of a single constraint \( C \) of the form \( ax + by \leq c \), where \( a, b, c \in \mathbb{Z}, a \neq 0 \). Prove that if \( x \) is bounds-consistent with respect to \( y \), then \( x \) is arc-consistent with respect to \( y \).

   (b) (1.5 points) Let us consider the CSP consisting of a single constraint \( C \) of the form \( ax + by = c \), where \( a, b, c \in \mathbb{Z}, a \neq 0 \). Is it true that if \( x \) is bounds-consistent with respect to \( y \), then \( x \) is arc-consistent with respect to \( y \)?

   If the answer is positive, prove so. If the answer is negative, give a counterexample.

2. (2 points) By using the variant of the simplex method you find most appropriate, solve the following linear program over real variables:

   \[
   \begin{align*}
   \text{min} & \quad 4x_{12} + 4x_{13} + 2x_{23} + 2x_{24} + 6x_{25} + x_{34} + 3x_{35} + 2x_{45} \\
   & \quad x_{12} + x_{13} = 20 \\
   & \quad -x_{12} + x_{23} + x_{24} + x_{25} = 0 \\
   & \quad -x_{13} - x_{23} + x_{34} + x_{35} = 0 \\
   & \quad -x_{24} - x_{34} + x_{45} = -5 \\
   & \quad -x_{25} - x_{35} - x_{45} = -15 \\
   & \quad 0 \leq x_{12} \leq 15 \\
   & \quad 0 \leq x_{23} \leq 10 \\
   & \quad 0 \leq x_{25} \leq 5 \\
   & \quad 0 \leq x_{13} \leq 8 \\
   & \quad 0 \leq x_{24} \leq 4 \\
   & \quad 0 \leq x_{34} \leq 15 \\
   & \quad 0 \leq x_{45} \leq 15
   \end{align*}
   \]

   Start with a basis that contains \( (x_{12}, x_{45}, x_{25}, x_{34}) \) in such a way that \( x_{13}, x_{35} \) and \( x_{24} \) are set to their upper bounds. Show in detail the initialization as well as all steps of every iteration (pricing, ratio test, update of the basic solution).

3. (2 points) Consider a system of linear constraints of the following form:

   \[
   Ax = b \\
   c^T x \leq d \\
   \ell \leq x \leq u, x \in \mathbb{Z}^m
   \]

   where \( A \in \{-1, 0, 1\}^{n \times m} \) is the vertex-edge incidence matrix of a directed graph \( G = (V, E) \), \( b \in \mathbb{Z}^n, c \in \mathbb{R}^m, d \in \mathbb{R} \) and \( \ell, u \in \mathbb{Z}^m \).

   Explain how the network simplex method can be used to solve this system efficiently.

4. (3 points) A problem of optimization modulo linear integer arithmetic consists in, given:

   - a formula \( F \) over linear integer arithmetic (i.e., a propositional formula where atoms are linear inequalities over the integers) defined over integer variables \( X = (x_1, \ldots, x_n) \), and
   - a cost function of the form \( c(X) = c_1x_1 + \ldots + c_nx_n \), where \( c_i \in \mathbb{Z} \), finding a model of \( F \) that minimizes \( c(X) \) over all models of \( F \) (or reporting \( F \) unsatisfiable).

   Assume that an SMT solver for linear integer arithmetic is available (source code too if necessary). To simplify, also assume that \( c(X) \) is non-negative over the models of \( F \), i.e., \( c(\sigma) \geq 0 \) for all \( \sigma : X \rightarrow \mathbb{Z}^n \) such that \( \sigma \models_{\text{LIA}} F \). Finally, assume that solving unsatisfiable instances can be much harder than solving satisfiable ones.

   Explain how to use or modify the available SMT solver so as to solve efficiently problems of optimization modulo linear integer arithmetic.