1. (3 points) Let \( G = (V, E) \) be an undirected graph. A feedback vertex set \( W \subseteq V \) of \( G \) is a set of vertices whose removal leaves a graph without cycles, i.e., the graph \( G_W = (V_W, E_W) \) where \( V_W = V - W \) and \( E_W = \{(u, v) \in E \mid u \notin W \text{ and } v \notin W\} \) is a disjoint union of trees.

(a) (2 points) Let \( G = (V, E) \) be the interaction graph of a binary CSP \( P \), and assume \( G \) has a feedback vertex set \( W \) of size \( k \). Describe an algorithm for solving \( P \) in time \( O(em^{k+2}) \) in the worst-case, where \( e \) is the number of constraints in \( P \), and \( m \) is the maximum size of the domains of the variables in \( P \). Assume the number of variables in \( P \) is \( O(e) \), and that evaluating a constraint takes time \( O(1) \).

(b) (1 point) Apply your algorithm from (a) to solve the following CSP \( P = (X, D, C) \):

\[
\begin{align*}
X &= \{x_i \mid 1 \leq i \leq 9\} \\
D(x_i) &= \{1, 2, 3\} \text{ for } 1 \leq i \leq 9 \\
C &= \left\{ \begin{array}{l} 
x_1 \geq x_3 + 2, \quad x_3 \neq x_5, \quad x_5 = x_7 + 1, \\
x_3 \neq x_4, \quad x_4 \neq x_5, \quad x_8 \mod x_5 = 1, \\
x_4 > x_2, \quad x_4 = 2x_6 + 1, \quad x_7 \leq x_8 - 2, \\
x_8 = x_9 \end{array} \right. 
\end{align*}
\]

2. (4 points) We want to schedule \( n \) tasks, i.e., for each task \( i \) (\( 1 \leq i \leq n \)), we want to determine its start time \( s_i \) (when the task starts) and its end time \( e_i \) (when the task ends). We are given the following input data:

- For each task \( i \) (\( 1 \leq i \leq n \)), its duration \( d_i > 0 \).
- A list of time constraints of one of the following two forms:
  
  (A) There should be at least \( k_{ij} \) units of time after task \( i \) ends before task \( j \) can start (for certain \( i \) and \( j \) such that \( 1 \leq i, j \leq n, i \neq j \)).
  
  (B) The difference (in absolute value) between the start times of tasks \( i \) and \( j \) should be of at most \( l_{ij} \) units of time (for certain \( i \) and \( j \) such that \( 1 \leq i, j \leq n, i \neq j \)).

The goal is to minimize the time span, i.e., to minimize \( \max_{1 \leq i \leq n} e_i - \min_{1 \leq i \leq n} s_i \).

(a) (1.5 points) Formalize this problem as a (real) linear program.

(b) (0.5 points) Show that the linear program cannot be unbounded.

(c) (2 points) Show that, if \( d_i, k_{ij}, l_{ij} \in \mathbb{Z} \) and the linear program is feasible, then we can ensure that the optimum solution produced by the simplex algorithm has integer values.

3. (3 points) A problem of linear boolean optimization consists in, given a boolean formula \( F \) defined over variables \( x_1, \ldots, x_n \), finding the model of \( F \) that minimizes over all models of \( F \) a function of the form \( c_1x_1 + \ldots + x_n \), where \( c_i \in \mathbb{Z} \) (or reporting that \( F \) is unsatisfiable).

Assume you have the code of a CDCL SAT-solver. Explain which changes you would make to the SAT solver in order to solve a problem of linear boolean optimization.