1. Let $G = (V, E)$ be an undirected graph. A feedback vertex set $W \subseteq V$ of $G$ is a set of vertices whose removal leaves a graph without cycles, i.e., the graph $G_W = (V_W, E_W)$ where $V_W = V - W$ and $E_W = \{(u, v) \in E \mid u \notin W \text{ and } v \notin W\}$ is a disjoint union of trees.

(a) (2 points) Let $G = (V, E)$ be the interaction graph of a binary CSP $P$, and assume $G$ has a feedback vertex set $W$ of size $k$. Describe an algorithm for solving $P$ in time $O(em^{k+2})$ in the worst-case, where $e$ is the number of constraints in $P$, and $m$ is the maximum size of the domains of the variables in $P$. Assume the number of variables in $P$ is $O(e)$, and that evaluating a constraint takes time $O(1)$.

(b) (1.25 points) Apply your algorithm from (a) to solve the following CSP $P = (X, D, C)$:

- $X = \{x_i \mid 1 \leq i \leq 9\}$
- $D(x_i) = \{1, 2, 3\}$ for $1 \leq i \leq 9$
- $C = \begin{cases} x_1 \geq x_3 + 2, & x_3 \neq x_5, & x_5 = x_7 + 1, \\ x_3 \neq x_4, & x_4 \neq x_5, & x_8 \mod x_5 = 1, \\ x_4 > x_2, & x_4 = 2x_6 + 1, & x_7 \leq x_8 - 2, \\ x_8 = x_9 & & \end{cases}$

2. (3 points) Assume we are given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, and let $S = \{x \in \mathbb{Z}^n \mid Ax = b, x \geq 0\}$. Assume also that we are given two vectors $c, d \in \mathbb{Z}^n$, and that there is $K > 0$ such that for all $x \in S$ and for all $1 \leq i \leq n$ we have $|x_i| \leq K$.

We are interested in computing $\max\{\max(c^T x, d^T x) \mid x \in S\}$. Model this as an Integer Linear Programming problem.

3. A binary clause is a clause with at most two literals. Given $S$ a set of non-empty binary clauses defined over the propositional symbols $p_1, \ldots, p_n$, the graph associated to $S$, denoted $G_S$, is the directed graph $G_S = (V, E)$, where $V = \{p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n\}$ and $E = \{(l, l') \mid \neg l \lor l' \in S\}$ (unit clauses, i.e. of the form $p$, are considered as $p \lor p$).

(a) (0.75 points) Show that if $(l, l') \in E$ then $(\neg l', \neg l) \in E$.

(b) (0.75 points) Show that if there is a path from $l$ to $l'$ in $G_S$, then $S \models \neg l \lor l'$.

(c) (0.75 points) What can you infer if there exists a literal $l$ such that there is a path from $\neg l$ to $l$?

(d) (0.75 points) What can you infer if there exists a propositional symbol $p$ such that $p$ and $\neg p$ belong to the same strongly connected component of $G_S$?

(e) (0.75 points) What can you infer if there exist literals $l$ and $l'$ (with different propositional variables) such that $l$ and $l'$ belong to the same strongly connected component of $G_S$?