Stability and non-stability of the FIFO protocol *

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Abstract

In this paper, we analyze the stability properties of the FIFO protocol in the Adversarial Queueing model for packet routing. We show a graph for which FIFO is stable for any adversary with injection rate $r \leq 0.1428$. We generalize this results to show upper bounds for stability of any network under FIFO protocol, answering partially an open question raised by Andrews et al. in [2]. We also design a network and an adversary for which FIFO is non-stable for any $r \geq 0.8357$, improving the previous known bounds of [2].

1 Introduction

First-in-First-out (FIFO) is one of the simplest queueing policies and has been used to provide best-effort services in packet-switched networks. One crucial aspect of FIFO's performance is that of stability, i.e. the question of whether there is a bound on the total size of packets in the network at all times.

The stability problem has been investigated under various models of packet routing, see for example [5, 6, 4, 7, 1]. The adversarial queueing model of Borodin et al. [3], was developed as a robust model of queueing theory in network traffic, and replaces stochastic by worst case inputs. Adversarial Queueing Theory considers the time evolution of a packet-routing network

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as a game between an adversary and a protocol. The adversary, at each time step, may inject a set of packets at some nodes. For each packet, the adversary specifies a network path that the packet must traverse, when the packet arrives to its final destination, it is absorbed by the system. If more than one packet wish to cross an edge e in the current time step, then FIFO resolves the conflict, maintaining a queue per edge. We shall say that a packet targets edge e if the edge belongs to its destination path, regardless whether that the packet path finishes at e.

A crucial parameter of the adversary is its *injection rate*. The rate of an adversary in this model, is specified by a pair (r, b) where $b \geq 1$ is a natural number and $0 < r < 1$. The adversary must obey the following rule:

> Of the packets that the adversary injects in any interval I, at most $\lceil r|I| \rceil + b$ can have paths that contain any particular edge.

Such a model allows for adversarial injection of packets that are "bursty".

The adversary, can select the initial configuration of the system. This is the main difference of our work with the previous model described in [2]. Without lost of generality, we will also assume that $b = 1$.

The motivation for study the behavior of packet communication networks, is to determine the conditions of *stability*, the fact that the number of packets in the system remains bounded, as the system dynamically evolves in time. Andrews et al. [2] solved several open questions posed in [3]. They also showed the existence of graphs and protocols that are not universally stable. In particular, they showed that for the network in Figure 2, the protocol FIFO is non-stable for values of $r \geq 0.85$. Later, Goel [8] proved that FIFO is not stable for the network in Figure 1. This result is a corollary of his main Theorem, showing that this network is not uniformly stable. However, the paper does not provide any upper bounds to r.

In [2], Andrews et al. proved the existence of a finite set of basic undirected graphs such that a graph G is stable for every r if and only if none of these graphs is a minor of G. Goel [8] presented three simple graphs H_1, H_2, H_3, which form this set for any directed graph. This result ensure the decidability of the question: *is G universally stable?* It remains an open problem the decidability of the following question: *is G stable for the FIFO protocol?* However, until now, it was not known if there is an $r_0 > 0$ such that, even these simple graphs, are stable for FIFO for any $r \leq r_0$.

A question raised at the end of [2], refers to the existence of a rate $r_0 > 0$ such that FIFO is stable against every adversary of rate $r \leq r_0$ and every network. In the first section of this paper, we partially answer in the positive their question, showing that for every network, FIFO is stable against any

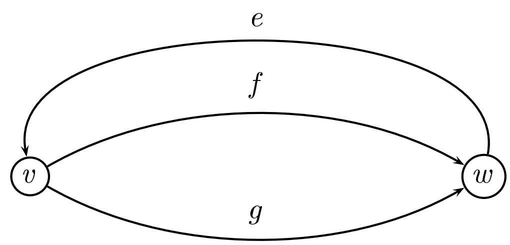

Figure 1: Goel's network $\mathcal{N}_1$.

adversary with a small injection rate, that depends on some parameters of the network. Before presenting the general result, we show that FIFO is stable for the particular network in Figure 1, for any adversary with rate $r \leq 0.1428$. We feel this proof will help to understand the general case.

Our second result is the design of a simple network and an adversary making the network unstable for $r \geq 0.8357$, thus improving the previous bounds in [2]. Our improvement is based on some new ideas of a) suitably exploiting "initial" paths and b) controlling the injection rounds in the construction.

For simplicity, we will omit floors and ceilings, and sometimes we will count steps and packets roughly. By carrying these through the computation we are loosing some additive constants but gaining in clarity. Given a network $\mathcal{N}$, and an edge $e \in \mathcal{N}$, $Q(e)$ we will denote the queue at e, and $e(t)$ will denote the size of $Q(e)$ at time t.

2 An upper bound for the stability for FIFO

Let us consider the network $\mathcal{N}_1$ in Figure 1. As already said, this network was considered in [8], to prove non-stability under any greedy protocol, in particular for the FIFO protocol, among other things. We prove here, that given any initial packet configuration for $\mathcal{N}_1$, and for every adversary $\mathcal{A}$ with injection rate $r \leq r_0$, for a given r_0 to be computed, the number of packets in the system remains bounded. So we can conclude that FIFO is stable for $\mathcal{N}_1$, under any adversary with rate $r \leq r_0$.

We will split the time into periods. In the first part of each period we analyze the evolution of the system from the initial time to the first step in which all the initial packets have arrived to their last queue. The second part allows enough time to guarantee that all initial packets have been served and that the system configuration reproduces the initial ones.

Notice that the in degree of edge e is two, therefore this edge can get an input flow bigger than any of the other two edges. So the largest queue builded in $\mathcal{N}_1$ must occur at e. We analyze the flow of initial packets with final destination e. The analysis of edges g and f can be done similarly.

Assume that at time 0, the system has a configuration of $f(0)$ packets in $Q(f)$, $g(0)$ packets in $Q(g)$ and $e(0)$ packets in $Q(e)$. Make $M = \max\{f(0), g(0), e(0)\}$. Let us run the system for M steps, which correspond to the first part of the period.

Let $P(t)$ denote the number of packets in the system at time t, let $P_e(t)$, $f_e(t)$ and $g_e(t)$ denote respectively the number of packets in $\mathcal{N}_1$, $Q(f)$ and $Q(g)$, that will pass through e. Then $P(0) = f(0) + g(0) + e(0)$ and $P_e(0) = f_e(0) + g_e(0) + e(0)$. As the protocol is FIFO, at time M, the number of packets from $P_e(0)$ which still have not been served by $Q(e)$ is at most $f_e(0) + g_e(0)$, and they are queued in $Q(e)$. At the same time, the maximum number of new injections of packets targeted with e is at most rM. Thus, $e(M) \leq rM + f_e(0) + g_e(0)$. Let

$$s = rM + f_e(0) + g_e(0), \tag{1}$$

then at time $M + s$ all the $P_e(0)$ old packets will have been served by $Q(e)$.

Consider the time intervals with duration: $s, rs, r^2 s, \ldots r^k s$, where k is such that $r^k s \geq 1$ and $r^{k+1} s < 1$. During these $k+1$ intervals of time, the total number of packets injected, targeted with e is

$$s \sum_{i=0}^{k} r^i s < sr \sum_{i=0}^{\infty} r^i = \frac{rs}{1-r}.$$

Let t_1 denote the time at the end of these intervals (which includes the initial time M). Then,

$$P_e(t_1) \leq \frac{r}{1-r} s + rM \tag{2}$$

By the previous definitions, $f_e(0) + g_e(0) \leq P_e(0) \leq P(0)$ and $M \leq P(0)$. Substituting in (1) we get $s \leq rP(0) + P(0) = (1+r)P(0)$. Plugging this bound for s and M into (2), we get

$$P_e(t_1) \leq \frac{r}{1-r}(1+r)P(0) + rP(0) \leq \left(\frac{2r}{1-r}\right) P(0).$$

To guarantee the stability of the system, it must be that the expression inside the parenthesis must be less than 1, which holds for a value $r_1 \leq 0.3334$.

On the other hand, at time t_1 the total number of packets in the system can be estimated in the following way: As all queues behave in the worst case as $Q(e)$, each queue can get at most one new packet per injection, and $\mathcal{N}_1$ has three queues, we get

$$P(t_1) \leq 3rM + \frac{3r}{1-r}s.$$

Substituting the value of s, we get

$$P(t_1) \leq 3rP(0) + \frac{3r}{1-r}(1+r)P(0) = \left(\frac{6r}{1-r}\right)P(0).$$

To guarantee the stability of the system, the quantity inside the parenthesis must be less than or equal to 1, which need a value $r_2 \leq 0.1428$.

Taking $r_0 = \min\{r_1, r_2\}$, we have shown that $P(t_1) \leq P(0)$ for any value of $r \leq r_0$.

Repeat the argument, getting an infinite time sequence $t_1, t_2, t_3, \cdots$. In the period between time 0 and t_1, the queues of the edges in $\mathcal{N}_1$ are bounded. As at each new period starting at t_{i+1}, the number of packets with a specific target, is a non-increasing function, we have proved,

Theorem 1. *For the network $\mathcal{N}$, given any initial configuration, FIFO is stable against any adversary with injection rate* $0 < r \leq 0.1428$.

The previous argument, can be extended to work in full generality, for any network $\mathcal{N}$ with k edges, maximum in-degree α, and maximum directed path length β,

Theorem 2. *For any network $\mathcal{N}$, given any initial configuration, there exist an $0 < r_{\mathcal{N}} < 1$, such that FIFO is stable with respect any adversary with injection rate $r \leq r_{\mathcal{N}}$, where $r_{\mathcal{N}}$ is independent of the initial configuration and it is a function of the maximum in-degree, the maximum directed path length and the number of edges.*

Proof. Assume that $\mathcal{N}$ has k edges, maximum in-degree α, and maximum directed path length β and that the injection rate of $\mathcal{A}$ is r. Notice that if $\alpha = 1$ then we have a tree or a ring, for which it is known that such a network is universally stable [2], so we assume $\alpha > 1$.

Let us denote the queues as $Q_1, Q_2, \ldots, Q_k$ and their loads at time $t \geq 0$ as $q_1(t), q_2(t), \ldots, q_k(t)$. Let $P(0) = \sum q_i(0)$ be the initial load.

We will construct an infinite sequence of time periods, t_i, at which $P(t_i) \leq P(0)$ thus keeping the net stable. Again, we will refer to the packets at time 0, as the *old packets*.

The fact that we are using a FIFO protocol implies that after a certain time all the old packets will leave the system. We will compute a bound to this time.

Consider the worst case of an old packet being last in a queue Q_j at time 0 and targeted with the largest simple path in the net. Rename the queues in this simple path as $Q_j \equiv Q_{j_0}, \ldots = Q_{j_{\beta-1}}$

Note that at time $M_1 = q_{j_0}(0)$ all packets of this queue will have been served. Thus these packets have passed to the next queues in the path. Moreover, they can be delayed by at most rM_1 new injections. Furthermore the size of any Q_{j_i} is bounded above by $(\alpha + r)M_1$.

We repeat the same procedure, each time considering the last queue in the path that still contains old packets. After $\beta - 2$ additional steps $(M_2, M_3, \ldots M_{\beta-1})$ all the old packets would disappear or being in Q_{j_β}.

Define $q(t) = \max_{i=0}^{k}\{q_i(t)\}$. Working in the previous way, an absolute bound for the delay of the last old packet in Q_j is $M = M_1 + \cdots + M_{\beta-1}$, where for every $0 < i < \beta$, we have $M_i \leq q(\sum_{j<i} M_j)$, with $M_0 = 0$. Moreover,during a period of $q(t)$ steps starting at time t, we have $q(t+q(t)) \leq (\alpha + r)q(t)$. Solving the recurrence, we have that the total time,

$$M \leq \sum_{i=0}^{\beta-1} (\alpha + r)^i q(0). \tag{3}$$

Consider consecutive time periods M, rM, r^2M, $\ldots$,r^lM, where l is such that $r^lM \geq 1$ and $r^{l+1}M < 1$. Let t_1 be the time at which r^lM finishes. The packets in $\mathcal{N}$ at time t_1 are all new, therefore the number of packets per queue is at most

$$rM + r^2M + \ldots + r^lM \ \leq \ rM \ + \ \frac{r}{1-r}M,$$

Therefore, $P(t_1) \leq \frac{2-r}{1-r}rkM$. Substituting the value of M from (3), we have

$$P(t_1) \leq \frac{2-r}{1-r} rk \left(\sum_{i=0}^{\beta-1} (\alpha + r)^i \right) P(0).$$

For the stability condition, we need $P(t_1)$ to be less than $P(0)$, which implies that we must choose an r such that,

$$\frac{2-r}{1-r} rk \left(\sum_{i=0}^{\beta-1} (\alpha + r)^i \right) \leq 1,$$

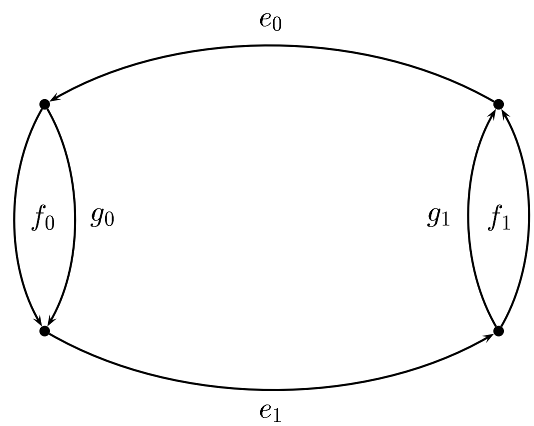

Figure 2: AAFKLL network $\mathcal{N}_2$.

which is equivalent to finding in the real interval $(0, 1)$, the root $r_{\mathcal{N}}$ of

$$-2Zk(\alpha+Z)^{(\beta+1)}+2Zk+Z^2k(\alpha+Z)^{(\beta+1)}-Z^2k+\alpha+2Z-1-\alpha Z-Z^2.$$

By the Bolzano's theorem, this polynomial has a root between 0 and 1, which is $r_{\mathcal{N}}$. $\square$

Using the polynomial obtained in the proof of the previous theorem, we can prove the stability under FIFO for any network. However, doing a particular analysis for specific networks, we can get better lower bounds for the values of the r's that make the network stable under FIFO protocol. For example, in [2] to prove the non-stability of the FIFO protocol, they use the network in Figure 2. Applying the previous techniques we get that the network is stable for $r \leq 0.0026732592$, while a direct proof gives a value for stability of $r \leq 0.03889$. In Section 3, we will use the network in Figure 3, to lower the injection rate for non-stability of FIFO. Applying again the techniques developed in Theorem 2, we get a value $r \leq 0.0003932323554$ for stability, while a more careful analysis will yield $r \leq 0.01062737157$.

3 An improved lower bound for instability of FIFO

In [2], for the network $\mathcal{N}_2$ in Figure 2 Andrews et al. gave a lower bound of 0.85, to prove the non-stability of FIFO. We slightly lower their bound to 0.8357, on the network that we consider in Figure 3.

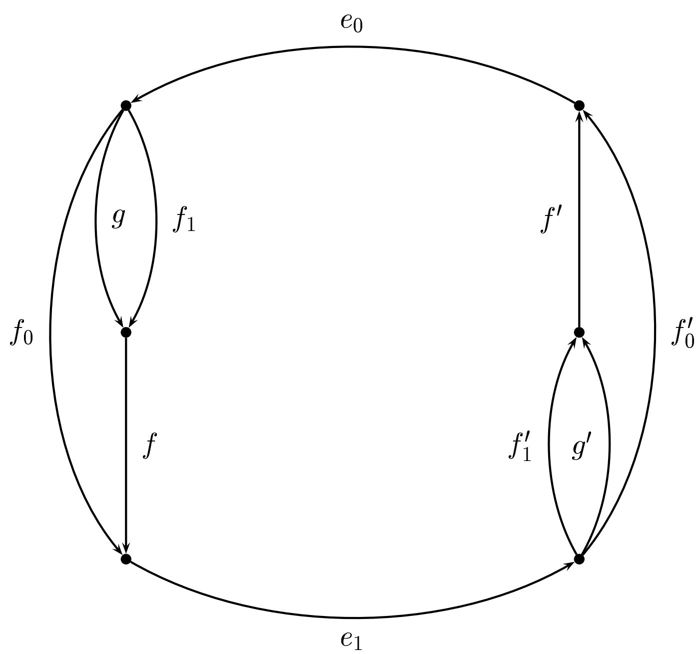

Figure 3: The network $\mathcal{N}_3$ for the non-stability of FIFO

Theorem 3. *There exists a network and an adversary $\mathcal{A}$, such that the protocol FIFO is unstable for $r \geq 0.8357$, starting from a non-empty configuration.*

Proof. Induction hypothesis: Let us consider the network $\mathcal{N}_3$ in Figure 3. At the beginning of the process, there are s packets in the queue $Q(e_0)$, targeted to go through $e_0 g f$. All the remaining queues are empty.

We will describe a policy $\mathcal{A}$ for an adversary with injection rate $r \geq 0.8357$, for which, at the end of a certain time (a phase), the system will have at the queue of e_1 $s' > s$ packets with destination $e_1 g' f'$. Each process will consist of 3 stages. We will use colors to keep track of the different kinds of injections. The s initial packets are blue.

Stage 1: During s steps, $\mathcal{A}$ injects into $Q(e_0)$, rs green packets with destination $e_0 f_1 f e_1 g' f'$. Also $\mathcal{A}$ injects into $Q(g)$, rs yellow packets targeted to g. Notice, that the blue packets will block the greens, and the yellow will delay the blues.

At the end of stage 1, the situation will be as follows: $Q(e_0)$ will contain rs blue packets, and $Q(g)$ will contain a mixture of rs yellow and blue packets. Let us compute how many of each color. If x denotes the number of blue packets remaining at $Q(g)$ after s steps, at the same period, there will rx yellow packets remaining at $Q(g)$, with $(1+r)x = rs$, from where we get $x = \frac{rs}{1+r}$ and therefore at the end of stage 1, we have $\frac{rs}{1+r}$ blue packets and $\frac{r^2 s}{1+r}$ yellow packets at $Q(g)$.

Stage 2: During rs steps, $\mathcal{A}$ injects into $Q(e_0)$, $r^2 s$ orange packets with destination $e_0 f_0 e_1 g' f'$; also $\mathcal{A}$ injects into $Q(f)$ $r^2 s$ purple packets with destination e_1. Notice that the orange packets are blocked by the greens.

At the end of stage 2, $Q(e_0)$ will contain $r^2 s$ green packets, and $Q(f)$ will contain $(r + \frac{1}{1+r})rs$ purple, blue and green packets. Let us compute the proportion of packets of each color that remains into $Q(f)$.

As the blue, green and purple packets arrive uniformly spread to $Q(f)$, in t steps there will enter into $Q(f)$, t green packets, rt purple packets and $\frac{t}{1+r}$ blue packets. Moreover,

$$t + rt + \frac{t}{1+r} = rs \Rightarrow t = \frac{(1+r)rs}{1+(1+r)^2}.$$

Therefore, the number of green packets remaining in $Q(f)$ at the end of stage 2 are

$$rs - \frac{rs(1+r)}{2+2r+r^2} = rs\frac{1+r+r^2}{1+(1+r)^2},$$

the number of purple packets remaining at the same time are

$$r^2 s - \frac{r^2 s(1+r)}{2+2r+r^2} = \frac{sr^2(1+r+r^2}{1+(1+r)^2},$$

and the number of blue packets are

$$\frac{rs}{1+r} - \frac{rs}{(1+r)}\frac{(1+r)}{2+2r+r^2} = \frac{rs(1+r+r^2)}{(1+r)(1+(1+r)^2)}.$$

Therefore, in $Q(f)$ the proportion of green packets is $\gamma = \frac{1+r}{1+(1+r)^2}$, the proportion of purple packets is $\delta = \frac{r(1+r)}{1+(1+r)^2}$ and the proportion of blue packets is $\beta = \frac{1}{1+(1+r)^2}$.

Stage 3: During $\left(r + \frac{1}{1+r}\right) rs$ steps, $\mathcal{A}$ injects into $Q(e_1)$, $\left(r + \frac{1}{1+r}\right) r^2 s$ red packets with destination $e_1 g' f'$. At the end of this final stage, $Q(e_1)$ will contain a total number of packets of

$$r^2 s + \frac{1+r}{1+(1+r)^2}\left(r + \frac{1}{1+r}\right) rs + \left(r + \frac{1}{1+r}\right) r^2 s - rs\left(r + \frac{1}{1+r}\right),$$

all of them waiting to go through edge e_1 and continue through $g'f'$.

If we compare s, the number of packets in $Q(e_0)$ at the beginning of the phase, and the number of packets in $Q(e_1)$ at the end of the phase, to assure non-stability we need,

$$\left(\frac{r(1+r)}{1+(1+r)^2}r(r-1)\right)\left(r+\frac{1}{1+r}\right)+r^2-1>0,$$

which implies that $r \geq 0.8357$. Therefore, we have proved the statement of the theorem.

□

Corollary 1. *At Stage 3, the time duration of* $\left(r+\frac{1}{1+r}\right)rs$ *steps is best possible, in the sense that any other smaller duration will yield higher lower bounds for the injection rate of* $\mathcal{A}$*.*

Proof. Let $r^2s+\alpha_r rs$ be the duration of round 3 in the proof of the previous theorem. We wish to choose the optimal α_r. Notice that $\Omega = rs(r+\alpha_r)$ packets leave $Q(f)$ during the interval. On the other hand, as input we have: $Y = r^2s$ orange packets; $Z = r^2s(r+\alpha_r)$ red packets and $X = \frac{1+r}{1+(1+r)^2}(r+\alpha_r)rs$ blue packets.

Therefore, we must have $X+Y+Z-\Omega > s$, that is

$$r^2+r^2(r+\alpha_r)+\frac{1+r}{1+(1+r)^2}(r+\alpha_r)r-r(r+\alpha_r)-1>0$$

$$r^2+r(r+\alpha_r)\left[\underbrace{\frac{1+r}{1+(1+r)^2}+r-1}_{E}\right]-1>0$$

Notice that the optimal value for α_r is obtained when we maximize X, since $E>0$ for $r>0.55$. Therefore the maximum α_r will take place when

$$\frac{1+r}{1+(1+r)^2}(r+\alpha_r)r \geq r\frac{r+r^2+1}{1+(1+r)^2},$$

which yields $\alpha_r \geq \frac{1}{1+r}$, proving the assertion of the corollary. □

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