Random scaled sector graphs*

Josep Díaz Jordi Petit Maria Serna

June 6, 2002

Departament de Llenguatges i Sistemes
Universitat Politècnica Catalunya
Jordi Girona Salgado 1–3
08034-Barcelona
{diaz,jpetit,mjserna}@lsi.upc.es

Abstract

In this paper, we introduce a new model of random graph, that we call random sector graph. This model aims to provide a tool for studying communication problems in networks of sensors using laser communication such as the ones addressed in the Smart Dust project. Current technology allows steering the laser cannon along a contigous sector, providing undirectional communication. Thus, random sector graphs are a generalization of random geometric graphs, in which this restricted communication is taken into account. We provide tight estimations of the maximum and minimum degree and show that random sector graphs are connected for an adequate selection of the sector radius. We also propose and analyze decentralized protocols to solve two basic issues in networks of sensors.

1 Introduction

This paper focuses on a new model of geometric graphs inspired by the Smart Dust project at the UC Berkeley [KKaP99, Sma]. It aims to provide a basic tool for the analysis of routing and transmission of information in a wireless ad-hoc communication network of sensors communicating through optical laser links.

Sensors are small (a few cubic millimeters) and cheap. Our motivating setting is to consider a randomly scattered large amount of sensors (from now on the *motes*) in a hostile environment, for instance a seismic fault or a volcanic area, and the objective is to pass information from the sensors to a more powerful computing unit, in charge of the monitoring system. The first communication technology for these networks used radio frequency, but due to size restrictions and the required low power operation, the new smart dust networks are planned to use free-space optical transmission. Free-space optical links have the limitation of an uninterrupted line-of-sight path for communication, but avoid radio interference.

In this work, we deal with a model where the sensor motes have a unique serial number (id) and are equipped with a clock, an optical device to receive and interpret laser beams, a short range laser with beam steering, a battery, a single sensor, a computing unit and a small memory. The motes communicate in an asynchronous way. The optical receiver can be

*Work partially supported by the IST Programme of the EU under contracts number IST-2001-33116 (FLAGS) and by the Spanish CICYT project and TIC-2001-4917-E.

oriented in the direction of the incoming light, and thus motes can receive information from any mote within a prescribed distance that looks at them [KKaP99]. To send information, they use an orientable laser beam, which current technology allows a range of forty degrees upward and sideward [Pis02].

In the monitoring scenario [KKaP99], the main goal of the sensor network is to forward information gathered by the network to some motes that are able to communicate with the exterior. This external computing unit will be responsible for processing the information. The motes are deployed in a terrain and should self-organize. Two basic network communication tasks are relevant: broadcasting and gathering.

We will deal with an ideal two dimensional squared region in which a mote falls with coordinates following the uniform distribution. The motes can orient its laser cannon in any position of a scanning area. This scanning area covers a (contiguous) sector of α degrees. The sector is oriented also randomly. We introduce a new model of random geometric graphs that we call *random scaled sector graphs*. To the best of our knowledge this is the first time that such a model has been considered. When $\alpha = 2\pi$, random scaled sector graphs correspond to an directed version of *random geometric graphs* [AR97a, AR97b, DPS98].

We address some basic graph parameters: degree and connectivity. We provide thight bounds for the maximum and minimum degree and show that random scaled sector graphs are almost strongly connected. We provide distributed algorithms for broadcasting and gathering, with random sources/sinks. We analyze them first for random geometric graphs. Then, we extend the result to random scaled sector graphs.

All through the paper, we use standard terminology for directed and undirected graphs and for probability. The distance $d(x, y)$ between two points in the plane will be the Euclidian distance.

2 Random scaled sector graphs

A mote can orient its laser beam in any position of a prescribed scanning area of α radians, however the mote can receive light from any point, within distance r, that is "looking" to it. This model gives raise to a directed graph model. Assume from now on that an angle α, in the interval $(0, 2\pi]$, is fixed.

Definition 2.1. *A graph is a* unit sector graph *if each vertex can be mapped to the center of a closed unit radius sector with amplitude α, in the plane, in such a way that there is an arc (u, v) if and only if v lies inside the sector associated to u.*

In the following we specify different sector orientations by an angle that gives the deviation from the horizontal axis. All our angle calculations will be made in arithmetic modulo 2π.

Definition 2.2. *Given a set V of n points in the unit square $[0, 1]^2$, a sequence $B = (b_1, \ldots, b_n)$ of angles and $r \in [0, 1]$, the corresponding* scaled sector graph *is the graph $\mathcal{G}_\alpha(V, B, r)$ where vertex i has as associated sector the portion of the disk with radius r delimited by angles b_i and $b_i + \alpha$ with respect to the horizontal axis anticlockwise.*

Let us recall the definition of geometric graphs.

Definition 2.3. *Given a set V of n points in the unit square $[0, 1]^2$ and $r \in [0, 1]$, the corresponding* geometric graph *is the graph $\mathcal{G}(V, r)$ where vertices i and j are connected if $d(i, j) \leq r$.*

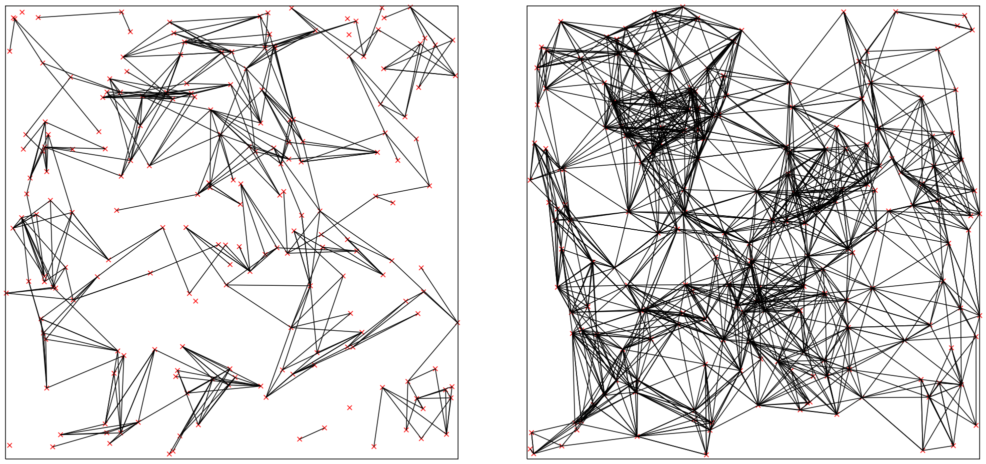

Figure 1: Two random sector graphs with $\alpha = 40^o$ and $\alpha = 2\pi$ for $n = 200$ and $r = 0.2$

A scaled sector graph is directed, but contains a natural undirected subgraph having the same vertex set and an edge $\{u, v\}$ whenever both (u, v) and (v, u) are arcs. Observe that if $\alpha = 2\pi$, any pair of vertices in a scaled sector graph have bidirectional communication, so that the associated undirected graph is a geometric graph.

Now we introduce the random model that we will use through the paper.

Definition 2.4. *Given a sequence* $X = (X_i)_{i\geq 1}$ *of* independently and uniformly distributed *(i.u.d.) random points in* $[0, 1]^2$*, a sequence* $B = (b_i)_{i\geq 1}$ *of i.u.d angles and a sequence* $(r_i)_{i\geq 1}$ *of numbers in* $[0, 1]$*. For any natural* n*, we write* $\chi_n = \{X_1, \ldots , X_n\}$ *and* $\beta_n = \{b_1, \ldots , b_n\}$*. We call* $\mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ *the* random scaled sector graph *with* n *nodes.*

Figure 1 shows two random sector graphs one with $\alpha = 40^o$ and the other with $\alpha = 2\pi$ (random geometric graph).

All through the paper we shall assume that the sequence of radii verify

$$r_n = \sqrt{\frac{a_n}{n}},$$

for some function $a_n = \Omega(\log n)$, for sufficiently large constant. The reason for this selection is to guarantee that the random geometric graph $\mathcal{G}(\chi_n, r_n)$ is connected with high probability (see [Pen99]), remaining sparse as r_n goes to zero. Recall that a sequence of events holds with high probability (w.h.p.) if the probability of the n-th event is at least $1 - O(1/n^c)$ for some $c \geq 1$.

Now we will introduce a particular type of scaled sector graphs that will be used along the paper. Consider a random sector graph $\mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ and let $\epsilon \in (0, 1)$ be a constant to be determined later. We dissect the unit square into $(\gamma r_n)^{-2}$ boxes, each of size $\gamma r_n \times \gamma r_n$ with

$$\gamma = \left(\left\lfloor \frac{2}{\epsilon r_n} \right\rfloor r_n\right)^{-1}.$$

Observe that $1/(\gamma r_n) = \left\lfloor \frac{2}{\epsilon r_n} \right\rfloor$ is an integer and that $\gamma \to \epsilon/2$ so that $\epsilon/4 \leq \gamma \leq \epsilon$ for n sufficiently large. We call such a dissection the *fine dissection*.

Definition 2.5 (Conservative graphs).
*A sector graph $\mathcal{G}_\alpha(V_n, B_n, r_n)$ is said to be ϵ-*conservative *if every box of the fine dissection contains at most $(1+\gamma)\gamma^2 a_n$ points and at least $(1-\gamma)\gamma^2 a_n$ points.*

An application of Chernoff bounds and Boole's inequality (see [DPS01]) yields the following result,

Lemma 2.1. *Let $\epsilon \in (0,1)$. Then, random scaled sector graphs $\mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ are ϵ-conservative w.h.p.*

Proof. Choose a box in the dissection and let Y be the random variable counting the number of points of χ_n in this box. As the points in χ_n are i.u.d.,

$$\mu = \mathbf{E}[Y] = \frac{n}{(\gamma r_n)^2} \sim \gamma^2 a_n.$$

Using Chernoff bounds,

$$\mathbf{Pr}\left[Y \geq (1+\gamma)\gamma^2 a_n\right] \leq \mathbf{Pr}\left[Y \geq \left(1+\frac{\gamma}{2}\right)\mu\right] \leq e^{-\left(\frac{\gamma}{2}\right)^2 \frac{\mu}{3}} \leq e^{-\frac{\gamma^4}{12} a_n}$$

and

$$\mathbf{Pr}\left[Y \leq (1-\gamma)\gamma^2 a_n\right] \leq \mathbf{Pr}\left[Y \leq \left(1-\frac{\gamma}{2}\right)\mu\right] \leq e^{-\left(\frac{\gamma}{2}\right)^2 \frac{\mu}{3}} \leq e^{-\frac{\gamma^4}{12} a_n}.$$

The number of boxes is smaller that n, so by Boole's inequality, the probability that for some box the number of points in the box is less than $(1-\gamma)\gamma^2 a_n$ or more than $(1+\gamma)\gamma^2 a_n$ is bounded by $2ne^{-\frac{\gamma^4}{12} a_n}$, which is $O(n^{-c})$ because $a_n = \Omega(\log n)$. ☐

3 Minimum and maximum degree

We will concentrate in results concerning points that are not very close to the boundary. We say that a point is *interior* if its distance to the boundary of $[0,1]^2$ is bigger than r_n, the set of interior points of a given graph G will be denoted by $I(G)$. Our first result gives sharp estimations of the expected maximum and minimum in and out degree of interior vertices.

Observe that in a random scaled sector graph, given two vertices x and y, within distance r_n, the pair (x,y) is an edge if y is inside the sector with radius r_n and α degrees. This property holds with probability $\frac{\alpha}{2\pi}$.

For a given vertex x, $d_{\mathsf{in}(x)}$ denotes the number of incoming edges and $d_{\mathsf{out}}(x)$ denotes the number of outgoing edges. For a graph G we define the maximum an minimum degrees as follows:

$$\delta_{\mathsf{in}}(G) = \min_{x \in I(G)} d_{\mathsf{in}}(x), \qquad \delta_{\mathsf{out}}(G) = \min_{x \in I(G)} d_{\mathsf{out}}(x),$$
$$\Delta_{\mathsf{in}}(G) = \max_{x \in I(G)} d_{\mathsf{in}}(x), \qquad \Delta_{\mathsf{out}}(G) = \max_{x \in I(G)} d_{\mathsf{out}}(x).$$

A sector spans over several boxes of the fine dissection. Consider a sector, the boxes that are completely in the sector are called the *entire boxes* and the boxes that are inside or cut by the sector are called the *touched boxes*.

In order to simplify notation, we define two auxiliary constants:

$$\kappa_1 = \kappa_1(\alpha) = (1-\gamma)\frac{\alpha - 4\alpha\gamma - 8}{2} \quad \text{and} \quad \kappa_2 = \kappa_2(\alpha) = (1+\gamma)\frac{\alpha + 4\alpha\gamma + 8}{2}.$$

One of the restriction for the selected ϵ in the definition of conservative graphs is that $\kappa_1(\alpha) > 1$.

Now we can state the first bound for conservative graphs.

Lemma 3.1. *Let $G_n = \mathcal{G}_\alpha(V_n, \beta, r_n)$ be a conservative sector graph. Then,*

$$\kappa_1 a_n \leq \delta_{\mathsf{out}}(G_n) \leq \Delta_{\mathsf{out}}(G_n) \leq \kappa_2 a_n.$$

Proof. Given any vertex u in $\mathcal{G}_\alpha(V_n, \beta, r_n)$, let S_u denote the sector associated to u. Let us bound the maximum number of boxes in the fine dissection, intersected by the perimeter of S_u. Notice that any segment with length r_n cuts at most $\frac{2}{\gamma}$ cells and any arc with radius r_n and angle α cuts at most $\frac{2\alpha}{\gamma}$ cells. Therefore, as the area of S_u is $\frac{\alpha}{2}r_n^2$, the total number of entire boxes is at least $\frac{\alpha}{2\gamma^2} - \frac{4+2\alpha}{\gamma}$. Moreover, the number of touched boxes is at most $\frac{\alpha}{2\gamma^2} + \frac{4+2\alpha}{\gamma}$. Multiplying by the number of points in the dissection we get the desired result. $\square$

Lemma 3.2. *Let $G_n = \mathcal{G}_\alpha(V_n, \beta, r_n)$ be a random conservative sector graph. Then, w.h.p.*

$$(1-\gamma)\kappa_1 a_n \leq \delta_{\mathsf{in}}(G_n) \leq \Delta_{\mathsf{in}}(G_n) \leq (1+\gamma)\kappa_2 a_n.$$

Proof. Let v be a vertex and C_v the disk with center v and radius r_n. The indegree of v is the number of vertices inside C_v, which look to v. The number of entire boxes of the fine dissection is at least $\frac{\pi}{\gamma^2} - \frac{4\pi}{\gamma}$, and the number of touching boxes is at most $\frac{\pi}{\gamma^2} + \frac{4\pi}{\gamma}$. Therefore, the number x_v of points in C_v is at least

$$(1-\gamma)\gamma^2 a_n \left(\frac{\pi}{\gamma^2} - \frac{4\pi}{\gamma}\right) = (1-\gamma)\kappa_1 a_n,$$

and at most

$$(1+\gamma)\gamma^2 a_n \left(\frac{\pi}{\gamma^2} + \frac{4\pi}{\gamma}\right) = (1+\gamma)\kappa_1 a_n.$$

As the probability that a vertex u in the circle looks at v is $\alpha/2\pi$, then $\mathbf{E}[d_{\mathsf{in}}(v)] = \frac{\alpha}{2\pi}x_v$. Using Chernoff's bounds,

$$\mathbf{Pr}[d_{\mathsf{in}}(v) \leq (1-\gamma)\kappa_1 a_n] \leq \mathbf{Pr}[d_{\mathsf{in}}(v) \leq (1-\gamma)\mathbf{E}[d_{\mathsf{in}}(v)]] \leq e^{-\frac{\gamma^2}{2}\frac{\alpha}{2\pi}x_v} \leq e^{-\frac{\gamma^2}{2}\frac{\alpha}{2\pi}\kappa_1 a_n},$$

and

$$\mathbf{Pr}[d_{\mathsf{in}}(v) \geq (1+\gamma)\kappa_2 a_n] \leq \mathbf{Pr}[d_{\mathsf{in}}(v) \geq (1+\gamma)\mathbf{E}[d_{\mathsf{in}}(v)]] \leq e^{-\frac{\gamma^2}{3}\frac{\alpha}{2\pi}x_v} \leq e^{-\frac{\gamma^2}{2}\frac{\alpha}{2\pi}\kappa_1 a_n}.$$

Using Boole's inequality, the probability that at least one vertex has degree outside the bounds is

$$\mathbf{Pr}[\delta_{\mathsf{in}}(G_n) \leq (1-\gamma)\kappa_1 a_n] \leq ne^{-\frac{\gamma^2}{2}\frac{\alpha}{2\pi}\kappa_1 a_n},$$

and

$$\mathbf{Pr}[\Delta_{\mathsf{in}}(G_n) \geq (1+\gamma)\kappa_2 a_n] \leq ne^{-\frac{\gamma^2}{3}\frac{\alpha}{2\pi}\kappa_1 a_n}.$$

And the probability of any one of these two events is $O(n^{-c})$ because $a_n = \Omega(\log n)$. $\square$

Combining Lemmata 2.1, 3.2 and 3.1, we obtain our main result in this section, sharp bounds for the minimum and maximum degrees of interior vertices in random sector graphs.

Theorem 3.1. *Let $G_n = \mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ be a random sector graph. Then, w.h.p.*

$$(1-\gamma)\kappa_1 a_n \leq \delta_{in}(G_n) \leq \Delta_{in}(G_n) \leq (1+\gamma)\kappa_2 a_n,$$

and

$$\kappa_1 a_n \leq \delta_{out}(G_n) \leq \Delta_{out}(G_n) \leq \kappa_2 a_n.$$

4 Connectivity

In this section we will show that random scaled sector graphs, with our selection of radius of r_n, are almost strongly connected w.h.p. That is, there is a directed path joining any pair of interior points. We use the following result on random geometric graphs.

Theorem 4.1 ([Pen99]). *Random geometric graphs $\mathcal{G}(\chi_n, r_n)$ are connected w.h.p.*

For a random scaled sector graph we will show the existence (w.h.p.) of an emulating directed path of length at most four, for any of the two orientations of an edge in the corresponding geometric graph.

Lemma 4.1. *Let $G_n = \mathcal{G}_\alpha(V_n, B_n, r_n)$ be a conservative sector graph. Then, w.h.p., given a pair (x, y) of interior points with $d(x, y) \leq r_n$, there is a directed path from x to y in G_n of length at most 4.*

Proof. If $(x, y) \in E(G_n)$, then we are done. Otherwise, we will show that, w.h.p., there is a directed path from x to y in $E(G_n)$.

Let C_x be the circumference with center x and radius r_n, notice that y is inside C_x. Let S_x be the sector associated to x, and let z_1 be the closest point in $S_x \cap C_x$ to the intersection of radius containing y with C_x. Let C_1 be the circumference with radius r_n and center z_1 and z_2 be the intersection of C_1 and C_x. Let C_2 be the circumference with radius r_n and center z_2 and z_3 be the intersection of C_2 and C_x. Finally, let C_3 be the circumference with radius r_n and center z_3 and let z_4 be the intersection of C_3 and C_x (see Figure 2).

Observe that all the points in S_x (or a portion of $\pi/3$ of it) are within distance r_n of the sector defined by z_1 and z_2. The points in this sector are within distance r_n of the sector defined by z_2 and z_3. The points in this last sector are within distance r_n of the sector defined by z_3 and z_4. Furthermore y must be inside one of the four sectors.

Assume without loose of generality that x and y are in the worst possible case, when the four sectors are needed. We will show the existence (w.h.p.) of three points y_1, y_2, y_3 one from each sector, so that the four arcs (x, y_1), (y_1, y_2), (y_2, y_3), (y_3, y) exist in G_n.

We can bound the probability that such a selection of connected vertices exists, taking into account that, in a conservative graph, each such sector has $c_1 a_n$ points for some constant c_1 and, the probability of having the arc (x, y) is $\frac{\alpha}{2\pi}$. So, such a path does not exists with probability at most

$$\left(\left(1 - \frac{\alpha}{2\pi}\right)^{c_1^2 a_n^2}\right)^4.$$

□

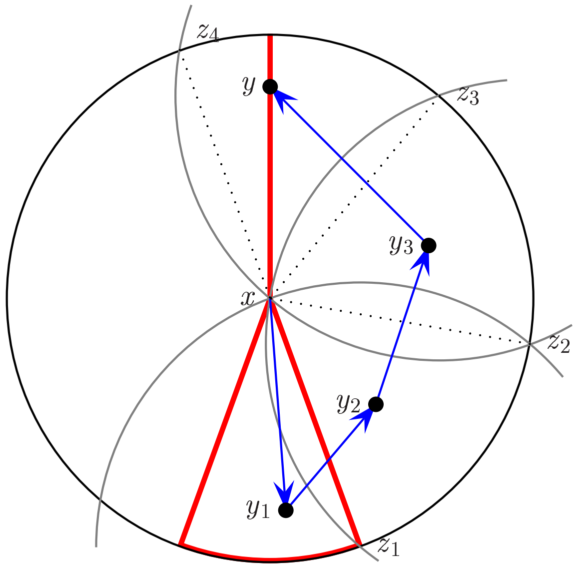

Figure 2: An emulating path

Combining the previous result with Theorem 4.1 we have,

Theorem 4.2. *For random scaled sector graphs $G_n = \mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$, w.h.p., for all $x, y \in I(G_n)$ there is a directed path from x to y.*

Proof. By Theorem 4.1 we know that w.h.p., the graph $\mathcal{G}(\chi_n, r_n)$ is connected. Using Boole's inequality and Lemma 4.1, the probability of an edge $\{x, y\}$ in $\mathcal{G}(\chi_n, r_n)$ failing to be emulated by a path in $\mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$, with length at most 4 (in one of the two possible orientations) is $O(n^{-c})$. □

5 Broadcasting

In this section we propose a broadcasting protocol and analyze its performance on random geometric graphs and random scaled sector graphs. We assume that a set of *source* motes, selected at randomly and independently, know an information that must be communicated to all the remaining motes in the network. We will obtain time and energy comsuption bounds. Observe that when we have only one mote acting as source the problem is the classical broadcasting problem.

The protocol simple-bro we propose is the following:

- Phase 0:
  - The source motes get level number 0. They perform a full scan, passing the message and level number 0. After that they go into an sleeping phase.
  - An awakened mote gets level 1.
- Phase i, $(1 \leq i)$
  - All motes with level i perform a full scan, sending the message and its level number incremented by one. After that they enter in a sleeping phase.

– Any mote that is awakened for the first time, gets level number i.

Ideally, the protocol should finish when all the motes get the message, we will guarantee time enough to run the protocol, so that w.h.p. all motes get the message.

First we analyze the protocol for random geometric graphs with random sources.

Theorem 5.1. *Let $p_n \in (0,1)$ and let $G_n = \mathcal{G}(\chi_n, r_n)$. Assume that a vertex in G_n becomes source with probability p_n, independently of all other vertices. Then, w.h.p., after performing $\left\lceil \sqrt{\frac{2\log n}{p_n a_n}} \right\rceil$ phases of the* simple-bro *protocol, all vertices in G_n know the message originated in the random sources.*

Proof. Let's dissect the unit square into $\frac{p_n n}{\log n}$ boxes. Each box has dimensions $\sqrt{\frac{\log n}{p_n n}}$. Using the coupon collector result, any square in this dissection will get w.h.p. a source provided the number of sources is roughly the number of boxes times its logarithm. But, for any given $\epsilon \in (0,1)$ and for big enough n,

$$\frac{p_n n}{\log n} \log\left(\frac{p_n n}{\log n}\right) = p_n n \left(1 + \frac{\log p_n - \log\log n}{\log n}\right) \leq (1-\epsilon)p_n.$$

Furthermore, using Chernoff's bounds the number of sources is at least $(1-\epsilon)pn$ w.h.p.

We further dissect each box of dimensions $\sqrt{\frac{\log n}{p_n n}}$ into squares of dimensions $\frac{r_n}{\sqrt{2}}$. Thus, the number of small boxes that fit in one dimension is

$$\sqrt{\frac{\log n}{p_n n}} \frac{\sqrt{2} r_n}{=} \sqrt{\frac{\log n}{p_n n}} \sqrt{\frac{2n}{a_n}} = \sqrt{\frac{2\log n}{p_n a_n}}.$$

Notice that any vertex in a box of dimensions $\frac{r_n}{\sqrt{2}}$, is connected to all vertices in its small box, and also to some vertices in neighboring boxes, except the corner points that will not reach the diagonal neighboring box. Therefore, the sources will cover in one step its small boxes, and some vertex in all but at least one neighboring small boxes. Continuing the flooding, after performing $\left\lceil \sqrt{\frac{2\log n}{p_n a_n}} \right\rceil$ phases, all the points in the big box, will collect the message. As the sources can initiate the broadcasting independently, we attain the desired result. □

Finally combining the above result with Lemma 4.1 we get

Theorem 5.2. *let $G_n = \mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ and let $p_n \in (0,1)$ be the probability that a mote becomes a source. Then, w.h.p., after performing $4\left\lceil \sqrt{\frac{2\log n}{p_n a_n}} \right\rceil$ phases of the* simple-bro *protocol, all nodes in G_n know the message originated in the random sources.*

Observe that the energy consumption per mote is that of a full scanning plus the energy lost in awakening and sleeping. This last part of the consumption can be avoided by precomputing an upper bound of the time to finish the broadcasting, so that the mote can program a sleeping period of known duration. In order to do that the motes will need synchronization, that can be achieved by incorporating the current time in each broadcast.

To avoid interferences some care must be taken in timing the scanning, one way to do it is by synchronizing the internal clock of all the motes as explained before. If the motes have a precomputed estimation of the time needed to perform a complete scanning of 2π degrees, they can agree on a period of time to perform a synchronized scanning in which at each time instant all active beams are oriented in the same direction.

6 Gathering

In this section we address the problem of gathering the information to be sent to the monitoring system. Now a set of *communicating* motes have a communication system that allows them to send information to a central computer base, the master. These motes will be responsible for collecting information from the network. The other motes will send their measurement and act as routers forwarding the information originated in motes farther to the communicating motes.

We propose a gathering protocol and analyze its performance on random geometric and random sector graphs. We assume the communicating motes are selected uniformly and independently at random. It is presumably that its number should be precomputed in advance as they might be more expensive. As before, we are interested in obtaining time and energy consumption bounds.

The protocol simple-gather we propose is the following:

- Phase 0:
  - All motes that are not communicating motes perform a full scan, passing its id.
  - Any mote x composes a message m_x with all the received id's and keeps it in its memory.
  - The communicating motes send $\langle x, m_x \rangle$ to the master.
  - The master identifies the id of all the received motes and split them so that any mote that can communicate in one hop gets an assigned communicating mote, in fact the angle to orient the laser to communicate with it.
  - This assignation is broadcasted through 4 phases of the simple-bro protocol, so that the motes that have been assigned a node can fix their laser beam in the adequate direction.
- Phase i, $(1 \leq i)$
  - All motes that have just fixed its laser direction, send a message $\langle x, m_x \rangle$ in their assigned direction.
  - All motes x that do have an assigned direction, compose a message with all the information received and send it.
  - The master identifies the id of all the received motes and split them so that any mote that can communicate in $i+1$ hops gets an assigned communicating mote.
  - This assignation is broadcasted through $4(i+1)$ phases of the simple-bro protocol, so that the motes that have been assigned a node can fix their laser beam in the adequate direction.

Notice that for a random geometric graph, the additional broadcasting in each phase is not needed, as the communication is bidirectional. The number of phases will be the same as for the simple-bro.

Lemma 6.1. *Let $G_n = \mathcal{G}_\alpha(\chi_n, \beta_n, r_n)$ and assume that a a mote can communicate with the master with probability p_n. Then, with high probability, after performing $4\left\lceil\sqrt{\frac{2\log n}{p_n a_n}}\right\rceil$ phases of the simple-gather protocol, all nodes in G_n have selected a neighbor to whom send information to be forwarded to the master.*

Proof. Using Theorem 5.1, for any randomly selected set of communicating nodes, we have that the broadcasting time is at most $\left\lceil \sqrt{\frac{p_n a_n}{2 \log n}} \right\rceil$. The `simple-trans` protocol attempts to recover this scheme orienting the edges from the leaves to the root. By Lemma 4.1 the number of phases is four times this depth. □

The energy in the above schema is bounded to a maximum of a complete scanning per phase and a laser positioning. As before, the scanning can be organized to minimize interference.

Once the spanning forest, directed to the communicating motes, is stablished, the network will enter into its final phase. In established periods of time, the measurement of motes is sent to the communicating motes and from them to the monitoring system. When this traffic starts motes will compose messages that contain the measurement of the motes directly connected to them, in the next step they will send the message forward to the next mote. As the flow comes closer to the source the size of the message is incremented, so it is of interest to get bounds for this maximum size.

In the case of geometric graphs the worst case arises when the communicating mote is in the center of the box. In such a case motes within distance k are inside a disk of radius k and outside a disk of radius $k-1$. Assuming a perfect distribution in the covered squares this gives a maximum number of single messages arriving at the same time to the communicating mote of the order $\sqrt{\frac{a_n \log n}{p_n}} r_n^2$ w.h.p. Therefore, we have an upper bound to the length of the longest message. Using the same emulations properties of Section 3, the same bound holds for random sector graphs.

The maximum length of the message can be reduced if all the motes know their coordinates. In such a case motes at the same level can compute a time interval, and forward messages only during this period. It should be notice that this might increase the gathering time, so that for critical networks the best solution should be to increase the number of communicating motes.

7 Conclusions and open problems

We have proposed a new model of geometric graphs to study algorithmic issues arising in the communication layer of networks of sensors. There are purely graph theoretical questions as, chromatic number, independent set, and others that remain to be studied. It is worth noticing that all the problems shown to be NP-complete for unit disk graphs in [CCJ90, DPPS01] remain NP-complete for unit sector graphs for any angle α. Furthermore the recognition problem is also NP-hard [BK98].

Our motes communicate through laser signal, and the objective of the network is the gathering of information to monitor the environment. We have assume that motes can communicate reliably, however due to land or metherological obstructions links between motes can fail. It will be of interest the generalization of the results in this paper to a model in which random vertex or edge faults are allowed.

The purpose of the Smart Dust is to make the motes sufficiently small so that they can fly with the wind. When the mobility is rare, the network may apply a periodical recomputation of the gathering tree. It is left for further study the modeling of the communication among motes under other levels of mobility. For highly mobile conditions, it seems that laser communication will not be desirable, due to the accuracy needed to orient the laser beam. In medium mobile conditions a throughly study of a dynamic version of random sectors graphs will be of interest.

References

[AR97a] M. J. Appel and R. P. Russo. The maximum vertex degree of a graph on uniform points in $[0,1]^d$. *Adv. Applied Probability*, 29:567–581, 1997.

[AR97b] M. J. Appel and R. P. Russo. The minimum vertex degree of a graph on uniform points in $[0,1]^d$. *Adv. Applied Probability.*, 29:582–594, 1997.

[BK98] H. Breu and D. G. Kirkpatrick. Unit disk graph recognition is NP-hard. *Computational Geometry. Theory and Applications*, 9(1-2):3–24, 1998.

[CCJ90] B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86:165–177, 1990.

[DPPS01] J. Díaz, M. D. Penrose, J. Petit, and M. Serna. Approximating layout problems on random geometric graphs. *Journal of Algorithms*, 39:78–116, 2001.

[DPS98] J. Díaz, J. Petit, and M. J. Serna. Random geometric problems on $[0,1]^2$. In J. Rolim, M. Luby, and M. Serna, editors, *Randomization and Approximation Techniques in Computer Science*, volume 1518 of *Lecture Notes in Computer Science*, pages 294–306. Springer-Verlag, Berlin, 1998.

[DPS01] J. Díaz, J. Petit, and M. Serna. A guide to concentration bounds. In S. Rajasekaran, P. Pardalos, J. Reif, and J. Rolim, editors, *Handbook on Randomized Computing*, volume II, chapter 12, pages 457–507. Kluwer Press, New York, 2001.

[KKaP99] J.M. Kahn, R.H. Katz and, and K.S.J. Pister. Mobile networking for smart dust. In *ACM/IEEE Int. Conf. on Mobile Computing and Networking*, pages 176–189, Seattle, 1999.

[Pen99] M. D. Penrose. On k-connectivity for a geometric random graph. *Random Structures & Algorithms*, 15(2):145–164, 1999.

[Pis02] K.S.J. Pister. Personal communication, 2002.

[Sma] Smart Dust Project. http://www.robotics.eecs.berkeley.edu/∼pister/SmartDust.