Chromatic number in random scaled sector graphs

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Abstract

Random scaled sector digraphs were introduced as a generalization of random geometric graphs, to model networks of sensors using optical communication. In this work, we study the value of the chromatic number $\chi(G_n)$, directed clique number $\omega(G_n)$, and undirected clique number $\widehat{\omega_2}(G_n)$ for random scaled sector graphs with n vertices, where each vertex spans a sector of α degres with radius $r_n = \sqrt{\frac{\ln n}{n}}$. We prove that for values $\alpha < \pi$, as $n \to \infty$ w.h.p., $\chi(G_n)$ and $\widehat{\omega_2}(G_n)$ are $\Theta(\frac{\ln n}{\ln \ln n})$, while $\omega(G_n)$ is $O(1)$, showing a clear difference with the random geometric graphs. For $\alpha > \pi$ w.h.p., $\chi(G_n)$ and $\widehat{\omega_2}(G_n)$ are $\Theta(\ln n)$, being the same for random sector and for random geometric graphs, while $\omega(G_n)$ is $\Theta(\frac{\ln n}{\ln \ln n})$.

1 Introduction

Massive networks of wireless sensors are call to play an important role in monitoring and disseminating information [ASSC02, AB02]. The general setting of such a network is to have a large collection of wireless sensors (motes) randomly scattered in a remote or hazardous terrain, performing tasks of distributed sensing. The sensing information gathered by the sensors should be relied to a base station. To communicate, between them or with a monitoring base, the sensors could use radio-frequency (RF) or optical communication. Inside the RF communication, sensors could use either an omnidirectional transmitter, which spreads the signal in a spherical region centered at the antenna, or a directional antennas, which have a focused beam spanning a sector of α degrees. In networks of sensors, directional antennas may have multiple advantages over omnidirectional antennas: less energy consumption, less fading area, furthermore as the transmission area is smaller the channel interference may have less influence [BGL02]. Sensors with optical communication can send information using an orientable laser beam embedded with an optical receiver, able to receive and interpret laser beams. Sensors can receive information from any mote within a prescribed distance that looks at them [KKRP99]. Many times, the sensors must be deployed randomly from some kind of vehicle.

In recent times, there has been an effort to provide a theoretical framework to study networks of sensors. For the omnidirectional RF communication network, a suitable model

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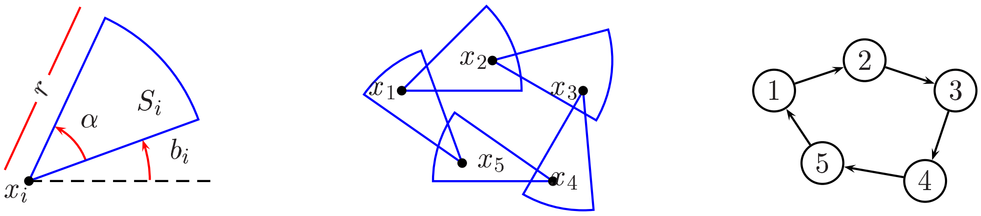


Figure 1: The sector of a sensor i and the communication between motes

is the *random geometic geometric*, also denoted *random scaled disk graph*. These graphs are the random scaled version of the *unit disk graphs* described in [CCJ90]. The model considers the network as a graph scaled into $[0,1]^2$, where the n random deployed sensors are the vertices of a random graph in $[0,1]^2$, and two vertices are connected if they are a within Euclidean distance r_n, corresponding to the broadcast range of the sensors. Many results are known about the properties of random scaled disk graphs. For instance, it is known that the chromatic number χ and the clique number ω of are asymptotically $\Theta(\ln n)$, when $r_n = O(\sqrt{f(n)}n$ with $f(n) = o(n)$ and $f(n) \to \infty$ (see for ex. Chapter 6 in [Pen03]).

The natural model for the case of directional RF and optical networks seems to be a generalization of the random geometric graph denoted the *random scaled sector graph*, which was introduced in [DPS03]. We refer the previous reference for the suitability of the model. In the setting under consideration, the sensors have a fixed constant angle α ($0 \leq \alpha < 2\pi$) of maximum scanning, defining a sector of transmission, moreover when sensor i falls in $[0,1]^2$, there will be an angle between the beam and the horizontal axis. Each mote i, is defined by three random variables x_i, y_i, β_i, where $0 < x_i, y_i < 1$ and $0 \leq \beta_i < 2\pi$. The two first coordinates fix the random position of i in $[0,1]^2$, while the third variable represents the random *elevation* of i with respect to the x-axis in $[0,1]^2$. We represent the beam emitted by i as the sector S_i, centered at i, with radius r, amplitude α and elevation β_i. Every other sensor which falls inside of S_i can receive the signal emitted by i (see Fig. 1). The random scaled sector graph is the digraph with vertices the sensors, in which there is an arc from i to j if j fells inside S_i, (see formal definition on Section 2). Some of the graph parameters for sector graphs coincide with the ones for geometric graphs, for instance for both graphs the threshold of the distance r to be connected is the same $r_n = \Theta(\sqrt{n/\ln n})$ (see [DPS03]). It should be noted, that in *practical applications*, the values of α are small,typically from $\pi/20$ to $\pi/4$, depending if the communication is RF or optical.

In this paper we study the value of the chromatic number $\chi(G_n)$, directed clique number $\omega(G_n)$, and undirected clique number $\widehat{\omega_2}(G_n)$ for random scaled sector graphs with n vertices and radius $r_n = \sqrt{\frac{\ln n}{n}}$. Asymptotically, we prove that for values $\alpha < \pi$, as $n \to \infty$ w.h.p., $\chi(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$, showing a clear difference with the geometric graphs. For $\alpha > \pi$, w.h.p. the value of $\chi(G_n)$ is $\Theta(\ln n)$ for both random sector and geometric graphs.

2 Results

A random scaled sector graph is defined in the following way,

Definition 1 ([DPS03]) *Assume that α is a fixed parameter of the sensors. Let $X = (x_i)_{i \geq 1}$ be a sequence of* independently and uniformly distributed *(i.u.d.) random points in $[0,1]^2$, let $B = (\beta_i)_{i \geq 1}$ be a sequence of i.u.d. angles and let $R = (r_i)_{i \geq 1}$ be a sequence of numbers in $[0,1]$. For any natural n, we write $X_n = \{x_1, \ldots, x_n\}$ and $B_n = \{\beta_1, \ldots, \beta_n\}$. We call the digraphs $G_n = \mathcal{G}_\alpha(X_n, B_n, r_n)$ the* random scaled sector graph *on n nodes, where $V(G_n) = X_n$ and the arcs are defined by: $(x_i, x_j) \in E(G_n)$ iff $x_j \in S_i$.*

We use the letter H to denote a subgraph of G_n. Δ, denotes the maximum degree of G_n. Given G_n, as usual the chromatic number, and the size of the maximum directed clique, are represented by $\chi(G_n)$ and $\omega(G_n)$, respectively. Since we are dealing with directed graphs, we introduce a new variable $\widehat{\omega_2}$, which represents the size of the maximum undirected clique, where for any two vertices $u, v \in V(G_n)$, to be members of the same undirected clique, only one of the two possible arcs, $(\overrightarrow{u,v})$ or $(\overrightarrow{v,u})$, need be present in the graph G_n. Thus, $\omega(G_n) \leq \widehat{\omega_2}(G_n) \leq \chi(G_n)$, and for $\alpha = 2\pi$, $\omega(G_n) = \widehat{\omega_2}(G_n)$.

We say G_n has a property T, *with high probability* (w.h.p), if as $n \to \infty$, we expect G_n to have property T, with probability $1 - O(1/n^c)$, for some $c > 0$. For other concepts and results in probability theory, look for example [Pen03].

In the remaining of the paper we give a proof for the following results, for the case when $r_n = \sqrt{\frac{\ln n}{n}}$,

Theorem 1 *Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, the size of the maximum directed clique, $\omega(G_n)$ is $\Theta(1)$. For $\pi + \epsilon < \alpha < 2\pi - \epsilon$, w.h.p., $\omega(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$.*

Theorem 2 *Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, w.h.p., the chromatic number, $\chi(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For $\pi + \epsilon < \alpha$, w.h.p., $\chi(G_n)$ is $\Theta(\ln n)$.*

Theorem 3 *Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, w.h.p., the size of the maximum undirected clique, $\widehat{\omega_2}(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For $\pi + \epsilon < \alpha$, w.h.p., $\widehat{\omega_2}(G_n)$ is $\Theta(\ln n)$.*

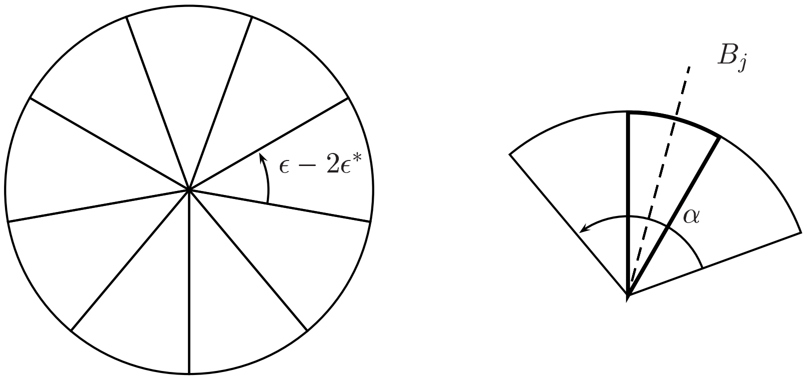
3 Basic constructions and lemmas

In this section, we present some tools and lemmas, which are needed to prove Theorems 1, 2, and 3. In order to lighten the notation we define the following variables,

$$a_n = \frac{\ln(n)}{\ln \ln(n)} \quad \text{and} \quad b_n = \sqrt{n \ln n}.$$

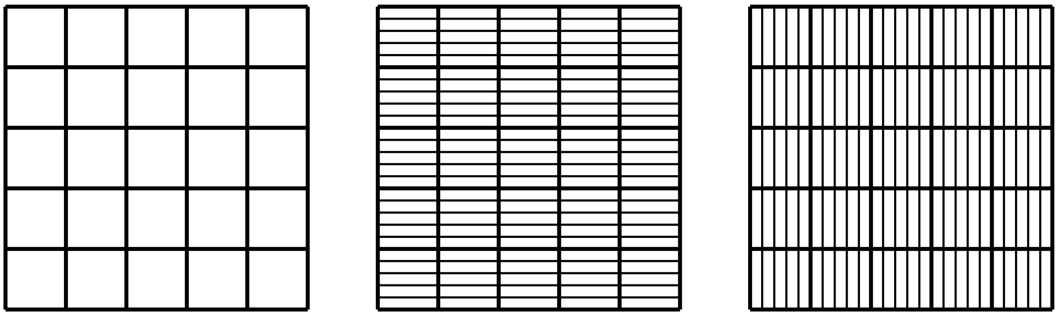
Notice that the orientation angle, β_i, of every mote i is drawn uniformly at random (u.a.r) from $(0, 2\pi]$. Let ϵ be a constant (depending on α), such that $\alpha = \pi + \epsilon$, for $0 < \epsilon < \pi$. We will consider a partition $\mathcal{B}$ of 2π into B classes, each of length $\epsilon - 2\epsilon^*$, with ϵ^* a constant chosen such that $\epsilon > 2\epsilon^*$ (see Figure 2). All motes i such that $\beta_i + \alpha/2$ fall within the same range will belong to the same class. More specifically, for any $1 < j \leq B$, the class B_j is defined as the class of mortes that their bisectrix falls between $(-\frac{3}{2} + j)\epsilon - (2j-3)\epsilon^*$ and $(-\frac{1}{2} + j)\epsilon - (2j-1)\epsilon^*$. Notice $B = \lceil \frac{2\pi}{\epsilon - 2\epsilon^*} \rceil$, so $B \in \mathbb{Z}$.

Throughout the paper, when we refer to the *dissection* $\mathcal{S}$ of $[0,1]^2$, we mean a partition of $[0,1]^2$ into $\frac{n}{\ln n}$ squares, each one of size $r_n \times r_n$.



(a) (b)

Figure 2: Angle partition for $\alpha > \pi + \epsilon$ (a) classes $\mathcal{B}$ (b) directions associated to a class B_j



(a) (b) (c)

Figure 3: The basic dissections of $[0,1]^2$ (a) $\mathcal{S}$ (b) horizontal subdivision (c) vertical subdivision

Lemma 1 ([DPS03]) *If n motes are distributed u.a.r. on $[0,1]^2$, w.h.p. each of the squares in the dissection $\mathcal{S}$ will contain $\Theta(\ln n)$ motes.*

The previous lemma was proved using Chernoff's bounds and Boole's inequality. Using the same techniques we can show

Lemma 2 *If n motes are distributed u.a.r. on $[0,1]^2$, w.h.p. every square in the dissection $\mathcal{S}$ will contain at most $3\ln n$ motes.*

For the following result we make use of an implication of Talagrand's inequality, as it is given in [MR00]:

Talagrand's Inequality *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $L_1, ..., L_m$, and satisfying the following for some b,r > 0:*

1. Changing the outcome of any one trial can affect X by at most b,
2. for any s, if X $\geq$ s there is a set of $< rs$ trials whose outcomes certify that X $\geq$ s.

Then, for any $0 \leq l \leq \mathbf{E}[X]$, $\mathbf{P}\left(|X - \mathbf{E}[X]| > l + 60b\sqrt{r\mathbf{E}[X]}\right) \leq 4e^{-l^2/8b^2r\mathbf{E}[X]}$.

Lemma 3 *Given the dissection $\mathcal{S}$ of $[0,1]^2$, divide each square of $\mathcal{S}$ in $\ln n$ rectangular regions of size $\frac{r_n}{\ln n} \times r_n$ (see Figure 3). Then, w.h.p. there exists at least one region R_i, which contains $(1-o(1))\frac{a_n}{B}$ motes from every class $B_j \in \mathcal{B}$.*

Proof Notice the number of regions in $[0,1]^2$is $\frac{n}{\ln n} \ln n = n$, therefore as the motes are distributed u.a.r. on $[0,1]^2$, by a *balls-and-bins* argument, there is a region R_i, which w.h.p. contains $a_n = \ln n / \ln \ln n$ motes. Let X_j be a random variable counting the number of motes in R_i which are in class B_j. Then $\mathbf{E}[X_j] = a_n/B$. Moreover, X_j is determined by the $m = (1-o(1))a_n/B$ trials specifying $\{\beta_1, \ldots, \beta_m\}$. Notice that changing the outcome of any one any β_l, $1 \le l \le m$, affect X_j by at most one, and also in order to certify $X_j \ge s$, only the outcomes of s trials (the s β_l's which fall in that class) are required. Thus the two conditions of Talagrand's inequality are satisfied with $b, r = 1$. By Talagrand's inequality and Boole's inequality, w.h.p. every class contains $(1-o(1))a_n/B$ motes. □

4 Proof of Theorem 1

4.1 $\alpha < \pi - \epsilon$

Proof When $\alpha < \pi - \epsilon$ the vertices of any clique must form a convex polygon. This can be proved by first noting that in every clique of size three, the three points cannot be collinear, and proceeding inductively. Since the sum of the angles of a convex polygon are $(|V|-2)\pi$, $\omega(G_n) \le \left\lfloor \frac{2\pi}{\pi - \alpha} \right\rfloor$. □

4.2 $\alpha > \pi + \epsilon$

Proof • First we establish the lower bound, by proving that a certain sufficient configuration of motes exists (w.h.p.). Consider the $\mathcal{S}$ partition of $[0,1]^2$. Further subdivide each small square into $\ln(n)$ equal (in terms of area) vertical regions (one can imagine drawing $\ln(n)$ equally spaced vertical lines). By Lemma 3, there is a region R_i containing w.h.p. $(1-o(1))\frac{a_n}{B}$ motes with orientation in the class B_1. Let M_1 be the set of such motes in R_i. Subdivide the region R_i in a_n/B cells, each cell a rectangle of width $\frac{1}{b_n}$ and height $\frac{Br_n}{a_n}$. Let Y be a random variable counting the number of cells containing at least one vertex from M_1, then $\mathbf{E}[Y] = (1-\frac{1}{e})a_n/B$ as $n \to \infty$, and as in the proof of Lemma 3, w.h.p, there are at least $(1-(\frac{1}{e}+o(1)))a_n/B$ cells containing at least one mote from M_1. To consider the worst case, assume a mote $m \in M_1$ is in the lower right-hand corner of a cell (see Fig. 4). Notice for any other mote m' to be connected with $m \in M_1$, it must be in the intersection of R_i with the sectors defined by Θ and Θ' with r_n. By trigonometry we get that m will have an arc with any mote in any cell more than distance (in either direction, up or down from m) $l = \left\lceil \frac{\cos((\alpha-\pi)/2)}{\sin((\alpha-\pi)/2)b_n} \right\rceil$.

However, notice that not all the motes in M_1 will have a the bisectrix exactly at 0, the orientation in B_1 has a range of $\epsilon - 2\epsilon*$, thus $l = \left\lceil \frac{\cos((\alpha-\pi-\epsilon+2\epsilon^*)/2)}{\sin((\alpha-\pi-\epsilon+2\epsilon^*)/2)b_n} \right\rceil$, as $\alpha > \pi + \epsilon$. For the worst case, when α assumes its lowest value, $l = \left\lceil \frac{\cos(\epsilon^*)}{\sin(\epsilon^*)b_n} \right\rceil$. For small x, $\sin(x) \sim x + O(x^2)$, given that ϵ^* is a constant, $l = c/b_n$, for some constant c. Since the height of each box is $\frac{Br_n}{a_n}$, w.h.p., $\omega(G_n) \ge c' a_n$, for sufficiently large constant c' dependent on α.

• Next we establish the upper bound, by showing that w.h.p., a certain necessary configuration cannot exist.

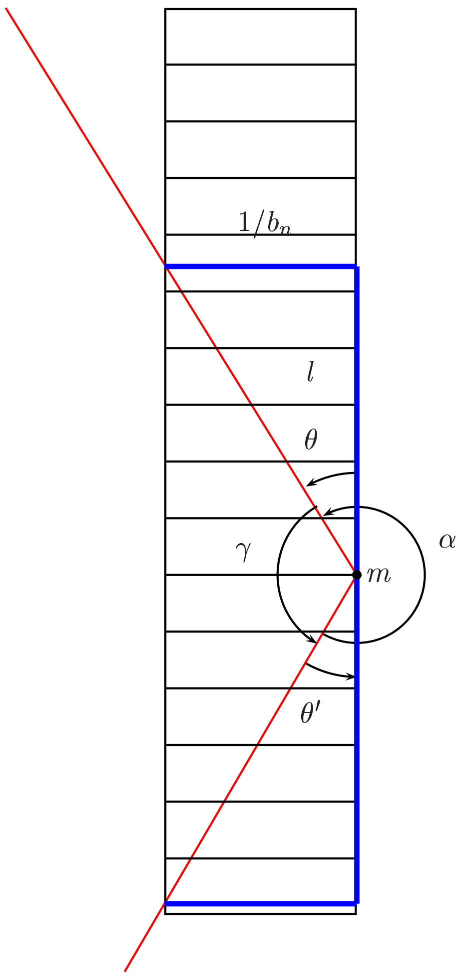


Figure 4: Proof of lower bound

Consider the partition $\mathcal{S}$ of $[0,1]^2$. Let ω^* represent the size of the largest directed clique in any square of $\mathcal{S}$, then the size of ω is upper-bounded by $9\omega^*$.

Again consider the $\mathcal{B}$-partition. Fix any square $S \in \mathcal{S}$. By Lemma 2, S contains at most $3\ln n$ motes. Select u.a.r. a_n motes from S. By the Pigeon-hole principle, at least $\frac{d}{B}a_n$ of those motes will have the bisectrix oriented into the same range, say class $B_j \in \mathcal{B}$. Let M_j be the set of all those motes. Define a partition of S into $\ln n$ strips in the following way: Create a fictional mote in S with $\frac{\alpha+2}{2} + \beta = (-\frac{1}{2} + i)\epsilon - (2i-5)\epsilon^*$, (the middle of B_j) draw a line with angle $\frac{\pi}{2}\frac{\alpha+2}{2} + \beta$ call it the *orientation* of the partition. Cover S with $\ln n$ rectangular strips parallel to the orientation (see Figure 5). In this partition of S, all the motes in the class M_j look in the same approximate direction, for these motes to be a part of the same clique every mote must see all the other motes along some specified direction.

We will prove that w.h.p., there exists a sufficiently large constant d such that no set of $\frac{d}{B}a_n$ motes form a clique and the statment of the theorem will follow.

As a first step, we will prove for d sufficiently large, any set of $\frac{d}{B}a_n$ motes, will (w.h.p.) occupy at least $(\ln n)^{9/10}$ strips. Assume the worst case, when the orientation of strips is parallel to the diagonals in S.

Any of the two diagonals in S have length $r_n\sqrt{2}$ and the surface of S is $r_n^2 = \ln n/n$. Each strip has width at most $\sqrt{2}r_n/\ln n = \sqrt{2}/\sqrt{n \ln}$. The area of the largest strip is bounded above by

$$\sqrt{2}r_n/\ln n \times r_n\sqrt{2} \leq \frac{2}{n} \ .$$

We wish to compute the probability that any set of $(\ln n)^{9/10}$ of the strips will contain all the $\frac{d}{B}a_n$ motes in S, which also are in class M_j.

The $(\ln n)^{9/10}$ strips have an area at most $\frac{2\ln n^{9/10}}{n}$. The area of any set of $(\ln n)^{9/10}$ strips divided by the area of S is at most $\frac{2}{(\ln n)^{1/10}}$, which is the probability that any given mote in S falls in the $(\ln n)^{9/10}$ strips.

Let p_1 be the probability that in any small square, a set of at least $\frac{d}{B}a_n$ motes falls in at most $(\ln n)^{9/10}$ strips. W.h.p. no small square has more than $3\ln n$ motes, thus the number of ways to choose a set of $\frac{d}{B}a_n$ motes from $3\ln n$ motes is

$$\binom{3\ln n}{\frac{d}{B}a_n} < n^3.$$

Moreover, as there are $n/\ln n$ small squares and at most n ways to choose $(\ln n)^{9/10}$ strips out of $\ln n$ strips,

$$p_1 \leq n^5 \left(\frac{2}{(\ln n)^{1/10}}\right)^{\frac{d\ln n}{B\ln\ln n}} \leq n^6 \left(\frac{1}{e^{\frac{\ln(\ln n)}{10}\frac{da_n}{B}}}\right) = n^6 e^{-\frac{d\ln n}{10B}}.$$

Therefore, as $n \to \infty$, a sufficiently large constrant d can be choosen so that, $p_1 \to 0$.

Given the above partition of S in $\ln n$ strips, ignore the first $\sqrt{\ln n}$ and last $\sqrt{\ln n}$ strips (we keep the middle, bigger ones). Every strip has width $\Theta(1/b_n)$. Define the *average height* of a strip as the average of the two sides of the strip. Let us consider again the worst case, when the difference in height between both sides of a strip is maximal, i.e. the case where the orientation of the partition is either $\pi/4$ or $3\pi/4$. Notice that the average height of all strips in the middle part will be larger that the average height of the smallest after the first

$\sqrt{\ln n}$ strips (strip T_i in Figure 5). Draw a diagonal line L of length $\sqrt{\frac{n}{n \ln n}}$, L spans $\sqrt{\ln n}$ of the discarded strips. The triangle with sides L, $L'/2$ and the edge of S is rectangle with two angles of $\pi/4$, so $L' = \Theta(\sqrt{\frac{\ln n}{n \ln n}}) = \Theta(\frac{1}{\sqrt{n}})$ In the same way, considering the triangle formed by $L + \Theta(1/b_n)$ and $L''/2 = \Theta(\frac{1}{2\sqrt{n}})$ together with the side of S, the average heigth of strip T_i is $\Theta(\frac{1}{\sqrt{n}})$, and the area of any middle strip is at least the area of T_i, which is $\theta(\frac{1}{\sqrt{n}\ln n} \times \frac{1}{\sqrt{n}} = \Theta(\frac{1}{n\sqrt{\ln n}})$.

Using the same argument used to show $p_1 \to 0$, we can find a sufficiently large constant d such that w.h.p., at least $\frac{d}{2B}a_n$ motes will fall outside of the first and last $\sqrt{\ln n}$ strips. Consider only these $\frac{d}{2B}a_n$ motes and label the motes along the specified direction, in the following way: Let $\gamma = \alpha/2 + \beta$ be the orientation of the strips, swap an immagianry line with sloope α through the $\ln -2\sqrt{\ln n}$ strips, w.h.p. at each instant the line will intersect one mote (motes are u.a.r. scatered). Labell the motes from m_1 to $m_{\frac{d}{2B}a_n}$, according to how soon they are scanned by the line. Partition the motes into pairs of sonsewqutive motes; motes m_{2i-1} and m_{2i} form a pair. Notice for each pair of motes, each of the two motes could be in the same strip or in different strips. Since we have $\Theta(\ln n)$ strips and $\frac{d}{4B}a_n$ mote pairs, by the pigeon hole principle, going along the specified direction, for d chosen sufficiently large at least $\frac{d}{8B}a_n$ motes in a pair, will be within $2 \ln \ln n$ strips of each other. Now in order for these motes to be part of the same clique, in every pair both motes must see each other. For $\gamma \in B_j$ this means that by design, every mote can see every other mote *above* it. Thus at least one of the necessary arcs is present. For the other arc to be present the upward-most mote (in the mote pair) must see the mote below. Since the strips in question have a width of at least $\Theta\left(\frac{1}{\sqrt{n}}\right)$, the horizontal coordinates of both points are drawn u.a.r. from $\left(0, \Theta\left(\frac{1}{\sqrt{n}}\right)\right]$. Thus in order to compute the probability of this event we will consider two disjoint cases.

Case one, the horizontal coordinates of at least one mote is in the interval $\left(0, \frac{1}{(\ln n)^{1/10}\sqrt{n}}\right]$. In that case we will assume with probability one, the upward mote sees the mote below. The probability of case one occurring is $\Theta\left(\frac{1}{(\ln n)^{1/10}}\right)$.

Case two, the horizontal coordinates of both motes are $> \frac{1}{(\ln n)^{1/10}\sqrt{n}}$. In this case since ϵ^* is a constant, the maximum area a mote sees of any strip which is within $2 \ln \ln n$ strips of it (in direction γ) is at most $\Theta(1/(na_n))$. This follows because the region of any one strip the mote sees has at most a width of $\Theta(r_n/a_n)$ and height $\Theta(1/b_n)$ (given the strip in question is within $2 \ln \ln n$ strips of the mote). Now the down-most mote in the pair, must fall in a strip. Also (since we are conditioning on being in case two), its horizontal coordinate is drawn u.a.r. from an interval of length $> \frac{1}{(\ln n)^{1/10}\sqrt{n}}$. Thus since every strip has height of $\Theta(1/b_n)$, conditioning on the particular strip the left-most mote falls into, the left-most mote falls u.a.r. into an area $> \Theta\left(\frac{1}{n(\ln n)^{6/10}}\right)$. Thus the probability the upward mote sees the downward mote is at most $\Theta\left(\frac{1}{(\ln n)^{1/10}}\right)$. Since for every pair of motes these events are independent of each other, the probability each pair of motes see each other is

$$\leq \Theta\left(\frac{1}{(\ln n)^{1/10}}\right)^{\frac{d \ln n}{8B \ln \ln n}} .$$

Let p_2 denote the probability in any square S in $\mathcal{S}$, there is a clique of size da_n or greater. Since (w.h.p.) in any square S we have at most n^3 sets of size da_n or greater, and $n/\ln n$

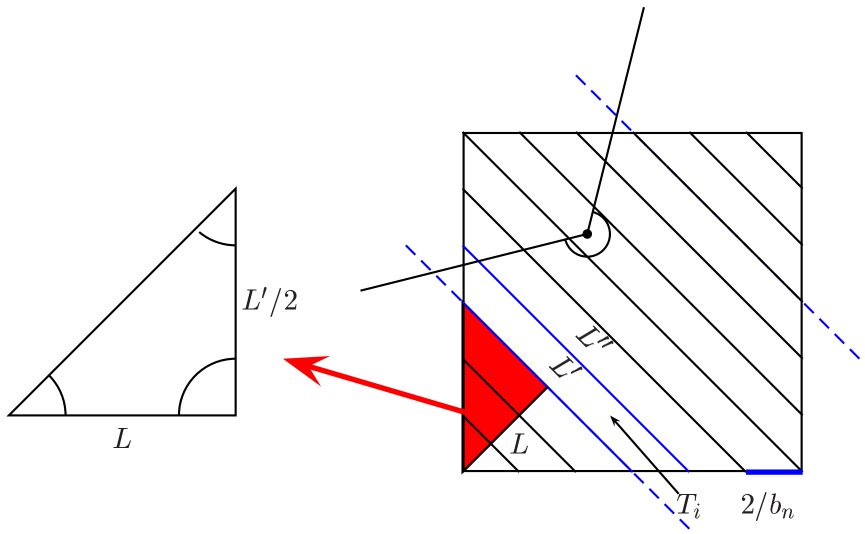


Figure 5: Partition of S by strips

squares in $\mathcal{S}$,

$$p_2 \leq n^4 \Theta\left(\frac{1}{(\ln n)^{1/10}}\right)^{\frac{d \ln n}{8B \ln \ln n}} \approx n^4 e^{-\frac{d \ln n}{80B}} \quad .$$

Therefore, there is a sufficiently large constant d such that $p_2 \to 0$.

$\square$

4.3 $\alpha = 2\pi$

For $\alpha = 2\pi$, the sector graphs became disks graphs, so the direction in the clique does not matter. Moreover, we know w.h.p., $\omega(G_n)$ is $\Theta(\ln n)$ [Pen03]. However, this fact can be directly verified, as above, by partitioning the $[0,1]^2$ unit square into $c\frac{n}{\ln n}$ regions, and bounding (w.h.p.) the number of motes in any region. Thus the value ω (for our particular of r) exhibits two transitions, one at $\pi + \epsilon$, the other at 2π.

5 Proof of Theorem 2

5.1 $\alpha > \pi + \epsilon$

Proof Dissect the unit square into $2n/\ln(n)$ smaller squares, each side equal to $r_n/\sqrt{2}$, let us call this partition $\mathcal{S}^*$. Observe that all the motes in this small square are at most a distance of r_n apart. By the pigeon hole principle at least one of the smaller squares has $\ln(n)/2$ motes. Consider this square which is a subgraph H. For each mote i in H with sector S_i, consider the sector $S_i^* = 2\pi - S_i$. It has an amplitud of $\pi - \epsilon$ (see Figure 6).

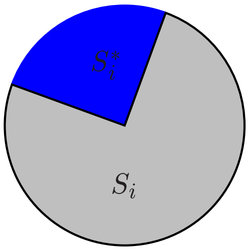


Figure 6: Sector S_i and complementary sector S_i^*

The sector which each mote does not see, it equals $\pi - \epsilon$. As it was the case for ω when $\alpha < \pi - \epsilon$, the motes of any independent set in H must form a clique in H^*, where H^* is the sector graph induced by S_i^*. Therefore as the amplitud is less than π, this set must form a convex polygon and $w(H^*) \leq \left\lfloor \frac{2\pi}{\alpha-\pi} \right\rfloor$. Let $\vartheta(H)$ represent the independence number of H. Then $\vartheta(H) = w(H^*) \leq \left\lfloor \frac{2\pi}{\alpha-\pi} \right\rfloor$. Using the fact that $\chi(G_n) \geq \chi(H) \geq V_H/\vartheta(H)$, we have $\chi(G_n) \geq \frac{\ln(n)}{2\lfloor \frac{2\pi}{\alpha-\pi} \rfloor}$. In order to establish the upper bound we use Brook's Theorem (see Lemma 1.3 in [MR00]): $\chi(G_n) \leq \Delta(G_n) + 1$. Form [DPS03], we know w.h.p., $\Delta(G_n)$ is $\Theta(\ln n)$. Thus using the results from above, we know, w.h.p. $\chi(G_n)$ is $\Theta(\ln n)$. □

5.2 $\epsilon < \alpha < \pi - \epsilon$

Proof

• First we establish the lower bound. Recall from above that $\widehat{\omega_2}(G_n) \leq \chi(G_n)$, where $\widehat{\omega_2}(G_n)$ is the size of the maximum undirected clique. Consider the partition $\mathcal{S}$ dissection of $[0,1]^2$. Further divide each square into $\ln(n)$ horizontal equal regions (see Figure 3). By Lemma 3, w.h.p. there is a region R_i which contains $(1 - o(1))a_n/B$ motes from each class in $\mathcal{B}$. consider the motes in R_1 that are oriented in B_1. Subdivide R_i into a_n/B rectangles, by drawing a_n/B evenly spaced vertical lines. Thus each rectangle has height equal to $1/b_n$ and width equal to $\frac{Br_n}{a_n}$. The expected number of rectangles containing at least one vertex in the limit as n $\to \infty$ is $(1 - \frac{1}{e})a_n/B$; and by Talagrands inequality w.h.p., there at least $(1 - (\frac{1}{e} + o(1)))a_n/B$ such rectangles. Assume a mote is in the upper (or lower) right-hand corner of a rectangle, and its $\beta = 0$, after a distance of $\left\lceil \frac{\cos(\alpha/2)}{\sin(\alpha/2)b_n} \right\rceil$ in either horizontal direction, the mote will be able to see a distance of $1/b_n$ in the vertical direction. That is after this distance the mote will have an arc with every other mote to its right within a distance of r_n in the rectangle in question. Repeating the same arguments as in the case of ω (which we omit in the interest of space), one can establish, w.h.p., $\widehat{\omega_2}(G_n)$ is at least da_n, where d is some constant dependent on α (see Figure 7).

• Next we establish the upper bound. Consider the dissection $\mathcal{S}$ on $[0,1]^2$. Let χ^* represent the largest chromatic number of any square S in $\mathcal{S}$, then $\chi(G_n)$ is upper-bounded by $9\chi^*$, as by Lemma 2, w.h.p. no square contains more than $3\ln n$ motes.

Again fix a square S in $\mathcal{S}$ and consider the partition $\mathcal{B}$. By the Pigeon hole principle, at least one class B_j will contain $\frac{d}{B}a_n$ motes in S all them oriented in almost the same direction. Let M_j be the set of all such motes. Define a partition of S into $\ln n$ strips in the following way: Create a fictional mote in S, whose bisectrix falls in $(-\frac{1}{2} + i)\epsilon - (2i - 5)\epsilon^*$. Next draw

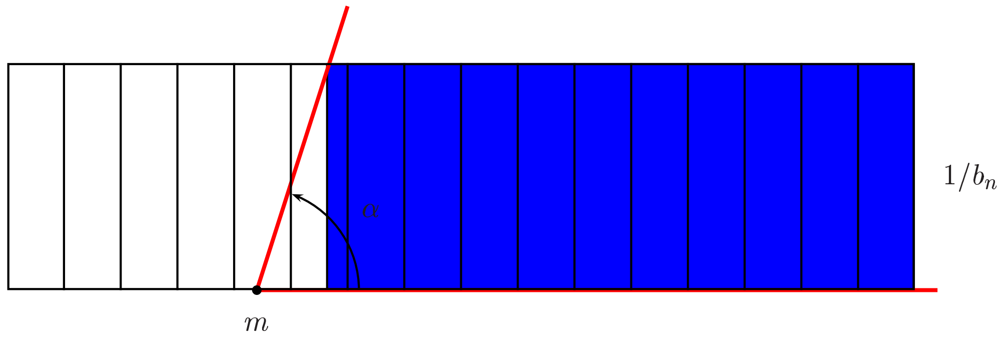


Figure 7: Figure for the proof of 5.2

a line perpendicular to the bisectrix of the fictional mote. This perpendicular line will be the *orientation* of the strips (note in the case of ω we used a different orientation). Partition S with $\ln n$ strips parallel to the orientation (see Figure 8 In this partition of S, all the motes in M_j look in the same *approximate* direction. We wish to prove that for a sufficiently large constant d, w.h.p. every set of $\frac{d}{B}a_n$ motes contains an independent set of size at least $1/3 \ln\ln n$.

Using similar arguments as in Section 1, we condition on the motes falling into the strips having average height $> \Theta\left(\frac{1}{\sqrt{n}}\right)$. Thus we have at least $\frac{d}{2B}a_n$ motes. Next, we will order these motes going along the specified direction. For example assume the specified direction is going left to right. Then we label the leftmost mote, 1, the second leftmost mote, 2, and so on. Next we partition the $\frac{d}{2B}a_n$ motes into $\frac{da_n}{2B\ln\ln n}$ classes. Each class C will contain $\ln\ln n$ motes. Again imagine that we are going from left to right, then class one will contain mote_1 to $\text{mote}_{\ln\ln n}$, and class two will contain the next $\ln\ln n$ motes. Now again by the pigeon hole principle at least one half of these classes occupy at most $2\ln n\frac{2B\ln\ln n}{da_n}$ strips. For d sufficiently large this means at least $\frac{da_n}{4B\ln\ln n}$ classes occupy at most $2\ln\ln n$ strips.

Now let us consider one class of these motes, say class one. And let us define two edges to be independent of each other if they have no endpoints in common. Thus the edges a-b and b-c are not independent, whereas the edges a-b and c-d are independent. First we prove the following: **Claim**: For any class C, if the largest independent edge set is less than $1/3\ln\ln n$, then there exits an independent set of size $1/3\ln\ln n$ or greater.

Proof of the Claim: This follows from the fact that the size of the vertex cover is at most 2 times the size of the maximal independent edge set with minimum cardinality. More specifically, assume the largest set of independent edges in C, is $\leq 1/3\ln\ln n$. Remove all the endpoints (along with any of their edges) from the graph. Since this was the largest independent set of edges (i.e. it is trivially maximal), any other edge not in this set must be dependent relative to some edge in this set (otherwise we would have included it in the set). Thus by removing all the endpoints in this set (the one with the most independent edges) we have deleted all the edges in this subgraph (i.e. in the class in question). Each independent edge has two endpoints. The largest such set is by assumption at most $1/3\ln\ln n$. Thus we have removed at most $2/3\ln\ln n$ motes (i.e. vertices). The class to begin with had $\ln\ln n$ motes. Thus we are left with at least $1/3\ln\ln n$ motes. And all the edges have been removed,

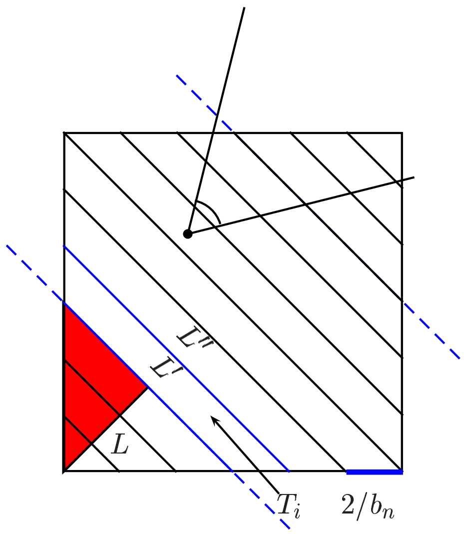


Figure 8: Partition of S in the prove 5.2

hence these $1/3 \ln\ln n$ motes form an independent set and the claim is proved.

Next let us compute the probability p_3 that in any class an independent edge set of size $1/3 \ln\ln n$ or greater exists. First we will restrict ourselves to classes S which occupy $2\ln\ln n$ strips or less (at least half of our classes are of this type). Thus every mote is within $2\ln\ln n$ strips of any other mote in the class. In order for two motes to share an edge, one mote must see the other going along the specified (in our case going from left to right) direction. Recalling the strips have average height of at least $\Theta\left(\frac{1}{\sqrt{n}}\right)$, the probability of this event is at most $\Theta\left(\frac{\sqrt{\ln n}}{a_n}\right)$. There are at most $(\ln\ln n)^2$ total edges in the class. The cardinality of the largest independent edge is $\leq 1/2 \ln\ln n$. Thus

$$p_3 \leq \Theta\left((\ln\ln n)^{\ln\ln n}\frac{\ln\ln n}{(\ln n)^{1/2}}\right)^{1/3\ln\ln n} \leq \frac{1}{(\ln n)^{4/10}}^{1/4\ln\ln n} \equiv e^{-\frac{d\ln n}{40B}} \quad .$$

Let p_4 denote the probability in any small square a set with da_n motes does not have an independent set of size $1/3 \ln\ln n$ or greater. There are at most (w.h.p.) n^3 ways to choose a set of da_n motes in any small square. We are considering $\frac{da_n}{4B\ln\ln n}$ classes of motes, i.e. all the classes which occupy at most $2\ln\ln n$ strips. No two classes have any motes in common, thus they are independent of each other. And we have $n/\ln n$ small squares, so

$$p_4 \leq n^4 e^{-(1/10(\ln\ln n)^2)\left(\frac{d\ln n}{4B(\ln\ln n)^2}\right)} \equiv n^4 e^{-\frac{d\ln n}{40B}} \quad .$$

Therefore, there exista a sufficiently large constant d such that $p_4 \to 0$.

Thus w.h.p., in every small square any set of da_n motes, has an independent set of size at least $1/3 \ln\ln n$. Now take any small square, and keep on picking independent sets of size

$1/3 \ln \ln n$. Assign all the motes in the same independent set the same color. When there are less than da_n motes left, assign all the remaining motes a different color. Thus we have colored all the motes in any small square (w.h.p.) with at most $\frac{3 \ln n - da_n}{1/3 \ln \ln n} + da_n$ colors. Since the chromatic number of the graph ($\chi(G_n)$), is at most a constant times this amount, w.h.p. $\chi(G_n) = O\left(\frac{\ln n}{\ln \ln n}\right)$. Combining with our lower bound, we have $\chi(G_n)$ is, w.h.p., $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. □

6 Proof of Theorem 3

6.1 $\alpha > \pi + \epsilon$

Proof First we prove that when $\alpha > \pi + \epsilon$, w.h.p., $\widehat{\omega_2} \geq \Theta(\ln(n))$. Again, consider the dissection $\mathcal{S}$ of $[0,1]^2$. By the pigeon hole principle at least one of the squares has $\ln(n)/2$ motes. Further all the motes in this square are at most a distance of r_n apart. Consider the subgraph H induced by the motes in this square S and consider the partition $\mathcal{B}$. The expected number of motes in any class $B_i \in \mathcal{B}$ is $\frac{\ln(n)}{2B}$. By Lemma 3, w.h.p. every class contains $(1-o(1))\frac{\ln(n)}{2B}$ motes. Next Divide S into $\ln(n)/2B$ stripes, by drawing $\ln(n)/2B$ evenly spaced vertical lines. The expected number of strips containing at least one vertex in the limit as $n \to \infty$ is $(1-\frac{1}{e})\frac{\ln(n)}{2B}$; and by Talagrand's inequality w.h.p., there at least $(1-(\frac{1}{e}+o(1)))\frac{\ln(n)}{2B}$ such strips. Consider the motes in B_1, going from left to right, every mote can see every other mote to its right (since $\alpha > \pi$). Thus, w.h.p, $\widehat{\omega_2}$ is at least $(1-(\frac{1}{e}+o(1)))\frac{\ln(n)}{2B}$.

For the upper bound, we know w.h.p., $\Delta(G_n) < \Theta(\ln n)$. □

6.2 $\alpha < \pi - \epsilon$

Proof We already established in our proof of $\chi(G_n)$ (see 5.2) a lower bound of $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For the upper bound, the proof is similar to that for $\omega(G_n)$ (see 4.2), and it is ommitted, we show that,w.h.p., $\widehat{\omega_2} \leq \Theta\left(\frac{\ln n}{\ln \ln n}\right)$. □

7 Conclusions and open problems

In this work, we produced asymptotic values for the directed clique $\omega^*(G_n)$, the modified clique $\widehat{\omega_2}(G_n)$ and the chromatic number $\chi(G_n)$ of random unit sector graphs. We observed that for the directed clique there are thresholds at $\alpha = 2\pi$ and at $\alpha = \pi$, but we have been unable to compute the value of ω^* for the particular value of $\alpha = \pi$. In the same manner, there are thresholds in the behavior of $\widehat{\omega_2}(G_n)$ and $\chi(G_n)$ at $\alpha = \pi$. Again, our methods do not seem to work for computing $\widehat{\omega_2}(G_n)$ and $\chi(G_n)$, for the particular value of $\alpha = \pi$ and the computation of $\widehat{\omega_2}(G_n)$, $\chi(G_n)$ and $\omega^*(G_n)$ at $\alpha = \pi$ remain open problems.

In the framework of channel assignment on random geometric graphs, McDiarmid [McD02] studied the value of the chromatic number for two types of random geometric graphs, sparse graphs, for which $r_n^2 n = o(\ln n)$ and $r_n^2 n = n^{o(1)}$ as $n \to \infty$. For those graphs, $\chi/\omega \to 1$ a.s. The second type of random geometric are the dense graphs, characterized by $r_n^2 n / \ln n \to \infty$ as $n \to \infty$. For those graphs, $\chi/\omega \to 1.103$a.s. McDiarmid leaves as open problem the chromatic number for random geometric graphs which are between dense and sparse, i.e. graphs for which $r_n^2 n / \ln n$ tend to constant c as $n \to \infty$. Notice that this radii is the one we are considering in this paper for defining the random scaled sector graphs. The work of McDiarmid suggested two open lines ; the asymptotic evaluation of $\chi(G_n)/\widehat{\omega_2}(G_n)$, when $\alpha < \pi - \epsilon$ and, the study of the behavior of $\chi(G_n)$ and $\widehat{\omega_2}(G_n)$ for the particular cases of *dense* and *sparse* random unit sector graphs.

To explicit more clearly the question $\chi(G_n)/\widehat{\omega_2}(G_n)$ for our model of random scaled sector graphs, we know via Talagrand's inequality that $\widehat{\omega_2}(G_n)$ is concentrated around it's mean. Since in the random scaled sector graph model, the size of the $\widehat{\omega_2}$ is a non-negative random variable Y determined by $m = 3n$ trials. In the same way as in the proof of Lemma 3, the conditions of Talagrand's inequality are satisfied with $b = 1$ and $r = 3$. Further we know that $\mathbf{E}[Y] \to \infty$. Thus applying Talagrand's inequality, the size of the Y is tightly concentrated around $\mathbf{E}[Y]$. In the same manner, we would like to show that the random variable Z denoting the size of $\chi(G_n)$ is concentrated around it's mean. In order to do so we need another implication of Talagrand's inequality, which we again state almost verbatim from [MR00] below.

Talagrand's Inequality *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $L_1, ..., L_m$. Let F be the event that for the outcome $x = (x_1, ..., x_m)$ of the trials, there exists a list of non-negative weights $b_1, ..., b_m$ such that:*

1. $\sum {b_i}^2 \leq D$; and
2. for any outcome y, we have $X(y) \geq X(x) - \sum_{x_i \neq y_i} b_i$,

then for any $0 \leq l \leq \mathbf{E}[X]$, $\mathbf{P}\left(|X - \mathbf{E}[X]| > l + 60\sqrt{D}\right) \leq 4e^{-l^2/8D} + 2\mathbf{P}\left(\overline{F}\right)$

As before, consider the dissection $\mathcal{S}$ on $[0,1]^2$. By Lemma 1, each square in $\mathcal{S}$ contains $\Theta(\ln(n))$ motes. Consider each of the squares separately and let H^* denote the square with the largest chromatic number. Then $\chi(G_n) \leq 9\chi(H^*)$. A random scaled sector graph can be generated via $m = 3n$ random trials, where for each vertex, two trials to specify the location and the third trial to specify the orientation. For each random scaled sector graph G_n, we make a list of $3n$ variables $b_1, ..., b_m$, where each b_i corresponds to one of the random trial, and define $b_i = 9$ if the $\left\lfloor \frac{i-1}{3} \right\rfloor$ vertex of G_n is also in $V(H^*)$, the and $b_i = 0$ otherwise. W.h.p., $\sum {b_i}^2 \leq \Theta(\ln(n))$. Further we know from Theorem 2, $\mathbf{E}[\chi(G_n)] > \Theta\left(\frac{\ln n}{\ln \ln n}\right)$. Thus we have the desired concentration. Unfortunately, these applications of Talagrand's inequality are not enough. Even though we can modify our proofs to compute the order of $\mathbf{E}[\chi(G_n)]$ and $\mathbf{E}[\widehat{\omega_2}(G_n)]$ we do not know if the leading constants in the order terms approach a limit as $n \to \infty$. Thus the question remains: does $\chi(G_n)/\widehat{\omega_2}(G_n) \to c$ as $n \to \infty$?

Other open question is the following, it is know that there exits a deterministic algorithm to find a max. clique within time $n^{4.5}$ for unit disks graphs [CCJ90]. It will be interesting to determine the complexity of finding a maximum directed or modified clique for unit scaled sector graphs.

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