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Authors: J. Diaz, D. Mitsche, X. Perez

Corresponding author: J. Diaz (diaz@lsi.upc.edu)

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Connectivity for Dynamic Random Geometric Graphs.

Josep Díaz, Dieter Mitsche, and Xavier Pérez-Giménez.

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I. INTRODUCTION

Random Geometric Graphs (RGG) have been a very influential and well-studied model of large networks, such as sensor networks, where the network nodes are represented by the vertices of the RGG, and the direct connectivity between nodes is represented by the edges. Informally, given a radius r, a random geometric graph results from placing a set of n vertices uniformly and independently at random on the unit torus $[0,1)^2$ and connecting two vertices if and only if their *distance* is at most r, where the distance depends on the chosen metric.

In the late 90's, Penrose [17], [18], Gupta and Kumar [12] and Appel and Russo [1] studied similar variations of this model, and gave accurate estimations for the smaller value of r at which, with high probability, a RGG becomes connected. This happens at the critical value $r_c = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$ for a RGG under the Euclidean distance in $[0,1)^2$, and in particular r_c is a *sharp threshold* for the connectivity of random geometric graphs. In fact, Goel et al. [9], proved that every monotone property of a RGG has a sharp threshold. Thereafter, many researchers have used those basic results on connectivity to design algorithms for more efficient coverage and communication in ad hoc networks (see e.g. [14]). On the other hand, much work has been done on the graph theoretical properties of *static* RGG, which is comprehensively summarized in the monograph of M. D. Penrose [19].

Recently, there has been an increasing interest for MANETs (mobile ad hoc networks). Several "practical" models of mobility have been proposed in the literature — for a survey of these models we refer to [15]. In all these models, the connections in the network are created and destroyed as the vertices move closer together or further apart. Many *empirical* results have been obtained for connectivity issues and routing performance and the different MANET models (see for example [20]). The paper [10] also deals with the problem of maintaining connectivity of mobile vertices communicating by radio, but from an orthogonal perspective to the one in the present paper: it describes a *kinetic data structure* to maintain the connected components of the union of unit-radius disks moving in the plane.

In this paper, we study a variation of the *Random Walk* model introduced by Guerin [11]. This model can be seen as the foundation for most of the mobility models developed afterwards (see [15]). The setting of the model that we study, is the following: Given an initial RGG with n vertices and a radius r set to be at the known connectivity threshold r_c, each vertex moves a distance s at every time step in some random direction. The initial direction of each vertex is chosen independently and uniformly at random from the interval $[0, 2\pi)$, and at every step each vertex updates its direction independently and with probability $1/m$. Therefore, each vertex moves in a particular direction for a geometrically distributed number of steps, and in average it travels a distance of $d = sm$ before changing direction. We denote this graph model the *Dynamic Random Geometric Graph*. Our choice of radius r_c is due to the fact that in many applications which are not life-critical, temporary network disconnections can be tolerated, especially if this goes along with a significant decrease of energy consumption [20]. This means, that the communication distance r should be kept as small as possible, but still large enough to guarantee a mostly connected graph, which happens for r around r_c.

For the case of static random geometric graphs, the connectivity thresholds for the torus $[0,1)^2$ and for the unit square $[0,1]^2$ are asymptotically the same (see for instance [19]). When talking about generic models of MANETS, most authors consider the unit square setting, where the vertices that touch the boundary of $[0,1]^2$, bounce back as a ball banging against a wall. From the experimental point of view, when doing simulations on large areas, the torus $[0,1)^2$ it seems to behave similarly as $[0,1]^2$ (see for ex. [4]). However, when using a rigorous analytic approach as the one done in this paper, the model on $[0,1]^2$ adds a greater degree of difficulty (the main problem is that at each step where one or more vertices touch the boundary, the probability space changes). We leave the connectivity on the unit square as an open problem (see Section IV).

Our main result (Theorem 1 in Section II) provides precise asymptotic results for the expected number of steps that the dynamic graph remains connected once it becomes connected, and the expected number of steps the graph remains discon-

The first and third authors were at the Department of Llenguatges i Sistemes Informàtics at the UPC, 08034 Barcelona. The second author was at the Institut für Theoretische Informatik, ETH Zürich, 8092 Zürich. Emails: {diaz, xperez}@lsi.upc.edu, dmitsche@gmail.com

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nected once it becomes disconnected. Our results are expressed in terms of n, s and m. Surprisingly, the final expression on the length of connectivity periods (asymptotically) does not depend on the expected number m of the steps between consecutive change of angles of a vertex (as long as the angles do eventually change, no matter how large the value of m is). It is worth to note here that the evolution of connectivity of this model is *not* Markovian, in the sense that staying connected for a large number of steps does have an impact on the probability of being connected at the next step. However, one key and rather counterintuitive fact is that, despite of this absence of the Markovian property, the argument to prove our result is mainly based on the analysis of the connectivity changes in two consecutive steps (see Lemma 9).

Throughout the paper, we consider the usual Euclidean distance on the unit torus $[0,1)^2$, but similar results can be obtained for any ℓ_p-normed distance, $1 \leq p \leq \infty$. Our results can also be extended to the k-dimensional torus $[0,1)^k$, for any fixed k.

To the best of our knowledge, the present work is the first one in which the dynamic connectivity of RGG is studied formally. In [6] the loosely related problem of the connectivity of the ad hoc graph produced by w vertices moving randomly along the edges of a $n \times n$ grid is studied. The authors of [16] use a similar model to the one used in the present paper to prove that if the vertices are initially distributed uniformly at random, the distribution remains uniform at any time. Further analytical work on path length durations in mobile ad-hoc networks and random walks in other models of dynamic random graphs was done in [13] and [2].

a) Notation and Organization.: Unless otherwise stated, all our results are asymptotic as $n \to \infty$. As usual, the abbreviation a.a.s. stands for *asymptotically almost surely*, i.e. with probability $1 - o(1)$ and u.a.r. stands for *uniformly at random*.

II. Known results on random geometric graphs, statement of the main result and outline of the proof

A. Random Geometric Graphs

We shall need some background about the known theory on random geometric graphs, which will be the starting point to study the dynamic case.

The formal definition of a random geometric graph is the following (see [19]): Given a set of n vertices and a positive real $r = r(n)$, each vertex is placed at some random position in the unit torus $[0,1)^2$ selected independently and uniformly at random (u.a.r.). We denote by $X_i = (x_i, y_i)$ the random position of vertex i for $i \in \{1,\dots,n\}$, and let $\mathcal{X} = \mathcal{X}(n) = \bigcup_{i=1}^n \{X_i\}$. Note that with probability 1 no two vertices choose the same position and thus we restrict the attention to the case that $|\mathcal{X}| = n$. We define $G(\mathcal{X}; r)$ as the random graph having $\mathcal{X}$ as the vertex set, and with an edge connecting each pair of vertices X_i and X_j in $\mathcal{X}$ at distance $\mathsf{d}(X_i, X_j) \leq r$, where $\mathsf{d}(\cdot,\cdot)$ denotes the Euclidean distance in the torus. We refer to $G(\mathcal{X}; r)$ as the *static model*.

Let K_1 denote the number of isolated vertices in $G(\mathcal{X}; r)$, which play an essential role in connectivity issues. Define the parameter $\mu = ne^{-\pi r^2 n}$ or reciprocally $r = \sqrt{\frac{\log n - \log \mu}{\pi n}}$. It is well known that the asymptotic behavior of μ characterizes the connectivity of $G(\mathcal{X}; r)$ (see e.g. [19] and Proposition 1 in [7]): if $\mu \to 0$, then a.a.s. $G(\mathcal{X}; r)$ is connected; if $\mu = \Theta(1)$, then a.a.s. $G(\mathcal{X}; r)$ consists only of isolated vertices and one giant component of size $> n/2$, and moreover, K_1 is asymptotically Poisson with parameter μ; if $\mu \to \infty$, then a.a.s. $G(\mathcal{X}; r)$ is disconnected. In this paper, we focus our attention on the case $\mu = \Theta(1)$ or equivalently $r = r_c = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$. Let us denote by $\mathcal{C}$ and $\mathcal{D}$ the events that $G(\mathcal{X}; r)$ is connected and disconnected respectively. Observe that, when $\mu = \Theta(1)$, the probability that $G(\mathcal{X}; r)$ is (dis)connected can be easily obtained:

$$\begin{aligned} &\mathbf{Pr}[\mathcal{C}] \sim \mathbf{Pr}[K_1 = 0] \sim e^{-\mu} \\ \text{and} \quad &\mathbf{Pr}[\mathcal{D}] \sim \mathbf{Pr}[K_1 > 0] \sim 1 - e^{-\mu}. \end{aligned} \tag{1}$$

A result that we will use in this paper is the fact that, for static random geometric graphs at the connectivity threshold r_c, the probability of having a component of size $\ell \geq 2$ different from the giant component is $\Theta(1/\log^{\ell-1} n)$. Moreover, a.a.s. those components are cliques contained in circles of small diameter [7].

B. Formal definition of the dynamic model

Given positive reals $s = s(n)$ and $m = m(n)$, consider the following random process $(\mathcal{X}_t)_{t\in\mathbb{Z}} = (\mathcal{X}_t(n,s,m))_{t\in\mathbb{Z}}$: At step $t = 0$, n vertices are scattered independently and u.a.r. over $[0,1)^2$, as in the static model. Moreover, at any time step t, each vertex i jumps a distance s in some direction $\alpha_{i,t} \in [0, 2\pi)$. The initial angle $\alpha_{i,0}$ is chosen independently and uniformly at random for each vertex i, and then at every step each vertex changes its angle independently with probability $1/m$. New angles are also selected independently and uniformly at random in $[0, 2\pi)$. Observe that the number of steps that each vertex must wait between two consecutive changes of angle has a geometric distribution with expectation m. Since the dynamic process is time-reversible, it also makes sense to consider negative steps. The dynamic random geometric graph is then defined as a sequence $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$, where for each particular value of t, $G(\mathcal{X}_t; r)$ is the random geometric graph with vertex set $\mathcal{X}_t$.

The case when s tends to 0 very fast is of special interest. In fact, given any $d = d(n) \in \mathbb{R}^+$, we can choose s arbitrarily small and m arbitrarily large such that $d = sm$, and the distance travelled by each vertex between two consecutive changes of angles is approximately exponentially distributed with mean $d = sm$. As a result, our model can be regarded as a discrete-time approximation of the following natural continuous-time counterpart, which we denote by $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{R}}$: the vertices move continuously at speed 1 around the torus rather than performing jumps at discrete steps, and each vertex changes direction according to an independent Poisson process of intensity $1/d$, thus the waiting time between two consecutive changes is exponential with mean d.

C. Main result

To state our main theorem precisely, we need a few definitions. We denote by $\mathcal{C}_t$ ($\mathcal{D}_t$) the event that $\mathcal{C}$ ($\mathcal{D}$) holds at step t. In $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$, define $L_t(\mathcal{C})$ to be the number of consecutive steps that $\mathcal{C}$ holds starting at step t (possibly ∞ and also 0 if C_t does not hold). The distribution of $L_t(\mathcal{C})$ does not depend on t (see Lemma 2), and we often omit the t when it is understood. $L_t(\mathcal{D})$ is defined analogously by interchanging $\mathcal{C}$ and $\mathcal{D}$ (in Lemma 11 it is shown that $L_t(\mathcal{C})$ and $L_t(\mathcal{D})$ are indeed random variables).

We are interested in the length of the periods in which $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected). More precisely, we consider the expected number of steps that $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected) starting at step t conditional upon the fact that it becomes connected (disconnected) precisely at step t:

$$\lambda_{\mathcal{C}} = \mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right) \qquad \text{and}$$
$$\lambda_{\mathcal{D}} = \mathbf{E}\left(L_t(\mathcal{D}) \mid \mathcal{C}_{t-1} \wedge \mathcal{D}_t\right).$$

Our main theorem then reads as follows:

Theorem 1. *Let $r = r_c$. The expected lengths of the connectivity and disconnectivity periods in $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ are as follows:*
If $srn = \Theta(1)$, then

$$\lambda_{\mathcal{C}} \sim \frac{1}{1-e^{-\mu(1-e^{-4srn/\pi})}}, \lambda_{\mathcal{D}} \sim \frac{e^{\mu}-1}{1-e^{-\mu(1-e^{-4srn/\pi})}}.$$

Otherwise, we have

$$\lambda_{\mathcal{C}} \sim \begin{cases} \frac{\pi}{4\mu srn} & \text{if } srn = o(1), \\ \frac{1}{1-e^{-\mu}} & \text{if } srn = \omega(1), \end{cases}$$

$$\lambda_{\mathcal{D}} \sim \begin{cases} \frac{\pi(e^{\mu}-1)}{4\mu srn} & \text{if } srn = o(1), \\ e^{\mu} & \text{if } srn = \omega(1). \end{cases}$$

Note that the results of $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ of both cases $srn = o(1)$ and $srn = \omega(1)$ correspond to the respective limits of the case where $srn = \Theta(1)$.

Intuitively speaking, the consequences of the result are the following. First observe that, asymptotically, the expected number of steps in a period of connectivity (disconnectivity) does not depend on how often the vertices of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ change their direction, since the expressions we obtained for $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ do not contain m. Moreover, $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ are non-increasing with respect to s, which corroborates the intuitive fact that having a big jump of the vertices at each step reduces the positive correlation existing between consecutive time steps for state $\mathcal{C}$ (or state $\mathcal{D}$). In particular, for $srn = \omega(1)$, $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ do not depend on s, since for such a large s the events of being (dis)connected at consecutive time steps are roughly independent. The case $srn = o(1)$ deserves some extra attention. Let us denote the expected total distance covered by each vertex during a connectivity period and a disconnectivity period by $\tau_{\mathcal{C}} = s \cdot \lambda_{\mathcal{C}}$ and $\tau_{\mathcal{D}} = s \cdot \lambda_{\mathcal{D}}$, respectively. In this case we have

$$\tau_{\mathcal{C}} \sim \frac{\pi}{4\mu rn} \sim \frac{\pi\sqrt{\pi}}{4\mu\sqrt{n\ln n}},$$
$$\tau_{\mathcal{D}} \sim \frac{\pi(e^{\mu}-1)}{4\mu rn} \sim \frac{\pi\sqrt{\pi}(e^{\mu}-1)}{4\mu\sqrt{n\ln n}},$$

which asymptotically do not depend on s. Note that these asymptotic relations still hold if s tends to 0 arbitrarily fast, as long as $s = o(1/(rn))$. In particular, this suggests that the related continuous-time model $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{R}}$ has a similar behaviour, and thus in that model the travelled distance during the periods of (dis)connectivity does not presumably depend either on the average distance $d = sm$ between changes of angle.

D. Overview of the Proof

The proof of the main result is structured into different lemmata, propositions and corollaries. The proofs of those partial results are highly technical. In this section we give the main waypoints to follow the proof.

The main ingredient of the proof is the fact that $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$ can be expressed in terms of the probabilities of events involving only two consecutive steps. Once more, we would like to stress this fact because the sequence of connected/disconnected states of $G(\mathcal{X}_t;r)$ is not Markovian, since staying connected for a long period of time makes it more likely to remain connected for one more step. More precisely, in Lemma 9 we show that it suffices to compute the probabilities of the events:

$$(\mathcal{C}_t \wedge \mathcal{D}_{t+1}), \quad (\mathcal{D}_t \wedge \mathcal{C}_{t+1}), \quad \mathcal{C} \quad \text{and} \quad \mathcal{D}. \tag{2}$$

However, the application of Lemma 9 requires that the expectations $\mathbf{E}\left(L_t(\mathcal{C})\right)$ and $\mathbf{E}\left(L_t(\mathcal{C})\right)$ are finite, which is proven in Lemma 11, using the Monotone Convergence Theorem. To obtain the probabilities of the events in (2), we start from Equation (1) in Subsection II-A and use Corollary 8, where we characterize the connectivity of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ at two consecutive steps. It turns out that the existence/non-existence of isolated vertices is asymptotically equivalent to the disconnectivity/connectivity of the graph, both in the static case $G(\mathcal{X};r)$ and for two consecutive steps of $G(\mathcal{X}_t;r)$. Proposition 6 characterizes the changes of the number of isolated vertices between two consecutive steps. The proof is based on the computation of the joint factorial moments of the variables accounting for these changes and using a well known theorem in probability (Theorem 1.23 in [3]), to characterize the fact that the random variables are Poisson. At first sight, it is not obvious that the probability of existence of components of larger sizes is negligible compared to the probability of sudden apperance of isolated vertices, but this is indeed shown in Lemma 7. The proof is quite technical and is split into five different cases, each case corresponding to a different type of components.

III. Proof of the Main Theorem

For the analysis of the dynamic model we need further definitions. We denote by $X_{i,t} = (x_{i,t}, y_{i,t})$ the position of

i at time t. Let $\mathcal{X}_t = \bigcup_{i=1}^{n}\{X_{i,t}\}$ be the set of positions of the vertices at time t. The following lemma (see [16]) indicates that the dynamic model at any fixed time t can be seen as a copy of the static model.

Lemma 2. *At any fixed step $t \in \mathbb{Z}$, the vertices are distributed over the torus $[0,1)^2$ independently and u.a.r. Consequently for any $t \in \mathbb{Z}$, $G(\mathcal{X}_t; r)$ has the same distribution as $G(\mathcal{X}; r)$.*

Let us consider two arbitrary consecutive steps t and $t+1$ of $(\mathcal{X}_t)_{t\in\mathbb{Z}}$, for an arbitrary fixed integer t (omitted from notation whenever it is understood). For each $i \in \{1,\dots,n\}$, the random positions $X_{i,t}$ and $X_{i,t+1}$ of vertex i at t and $t+1$ are denoted by $X_i = (x_i, y_i)$ and $X_i' = (x_i', y_i')$. Let also $\mathcal{X} = \mathcal{X}_t$ and $\mathcal{X}' = \mathcal{X}_{t+1}$. If $2\pi z_i$ ($z_i \in [0,1)$) is the angle in which i moves between t and $t+1$, then $x_i' = x_i + s\cos(2\pi z_i) \mod 1$ and $y_i' = y_i + s\sin(2\pi z_i) \mod 1$ (hereinafter, the notation $\mod 1$ will be often omitted for simplicity). That motivates the following description of the model at t and $t+1$ in terms of a three dimensional placement of the vertices, in which the third dimension is interpreted as a normalized angle: For each $i \in \{1,\dots,n\}$, define the random point $\widehat{X}_i = (x_i, y_i, z_i) \in [0,1)^3$, and let $\widehat{\mathcal{X}} = \bigcup_{i=1}^{n}\{\widehat{X}_i\}$. By Lemma 2, all random points $\widehat{X}_i$ are chosen independently and u.a.r. from the 3-torus $[0,1)^3$. Moreover, $\widehat{\mathcal{X}}$ encodes all the information of the model at steps t and $t+1$: If we map $[0,1)^3$ onto $[0,1)^2$ by the following surjections

$$\pi_1 : (x,y,z) \mapsto (x,y)$$
$$\pi_1 : (x,y,z) \mapsto (x + s\cos(2\pi z), y + s\sin(2\pi z)),$$

we can recover the positions of vertex i at times t and $t+1$ from $\widehat{X}_i$ and write $X_i = \pi_1(\widehat{X}_i)$ and $X_i' = \pi_2(\widehat{X}_i)$.

For any measurable set $\mathcal{A} \subseteq [0,1)^2$, the events $X_i \in \mathcal{A}$ and $X_i' \in \mathcal{A}$ are respectively equivalent to the events $\widehat{X}_i \in \pi_1^{-1}(\mathcal{A})$ and $\widehat{X}_i \in \pi_2^{-1}(\mathcal{A})$ in this new setting. Furthermore, by setting $\mathcal{A}_z = \mathcal{A} - (s\cos(2\pi z), s\sin(2\pi z))$ we get

$$\mathsf{Vol}(\pi_2^{-1}(\mathcal{A})) = \int_{[0,1)} \Big(\int_{\mathcal{A}_z} dxdy\Big)\, dz = \mathsf{Area}(\mathcal{A}).$$

In addition, observe that $\mathsf{Vol}(\pi_1^{-1}(\mathcal{A})) = \mathsf{Vol}(\mathcal{A} \times [0,1)) = \mathsf{Area}(\mathcal{A})$, and hence we have

$$\mathsf{Area}(\mathcal{A}) = \mathsf{Vol}(\pi_1^{-1}(\mathcal{A})) = \mathsf{Vol}(\pi_2^{-1}(\mathcal{A})). \tag{3}$$

In view of Lemma 2, for any measurable sets $\mathcal{A} \subseteq [0,1)^2$ and $\mathcal{B} \subseteq [0,1)^3$, we have $\mathbf{P}(X_i \in \mathcal{A}) = \mathsf{Area}(\mathcal{A})$, $\mathbf{P}(X_i' \in \mathcal{A}) = \mathsf{Area}(\mathcal{A})$ and $\mathbf{P}(\widehat{X}_i \in \mathcal{B}) = \mathsf{Vol}(\mathcal{B})$, which is compatible with (3).

For each $i \in \{1,\dots,n\}$, consider $\mathcal{R}_i = \{X \in [0,1)^2 : \mathsf{d}(X, X_i) \le r\}$ and $\mathcal{R}_i' = \{X \in [0,1)^2 : \mathsf{d}(X, X_i') \le r\}$. Let $\widehat{\mathcal{R}}_i = \pi_1^{-1}(\mathcal{R}_i)$ and $\widehat{\mathcal{R}}_i' = \pi_2^{-1}(\mathcal{R}_i')$ be their counterparts in $[0,1)^3$. Note that vertex j is connected to vertex i at time t iff $\widehat{X}_j \in \widehat{\mathcal{R}}_i$. Thus, X_i is isolated in $G(\mathcal{X}; r)$ iff $(\widehat{\mathcal{X}} \setminus \{\widehat{X}_i\}) \cap \widehat{\mathcal{R}}_i = \emptyset$, and analogously X_i' is isolated in $G(\mathcal{X}'; r)$ iff $(\widehat{\mathcal{X}} \setminus \{\widehat{X}_i\}) \cap \widehat{\mathcal{R}}_i' = \emptyset$.

The following technical lemma is needed in several places. It gives elementary bounds on the volume of the intersection of two regions as a function of the distance of the corresponding points and the stepsize. Note that part 1) and part 2) can easily be described in two dimensions, but since part 3) and part 4) are better explained in three dimensions, we use the third dimension throughout.

Lemma 3. *Assume $\mu = \Theta(1)$. There exists a constant $\epsilon > 0$ such that for large enough n the following statements are true: For any $i, j \in \{1,\dots,n\}$ (possibly $i = j$),*

1) *if $d(X_i, X_j) > r$ then $\mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}_j) \le \frac{\pi}{2} r^2$,*
2) *if $s < r/7$ and $d(X_i, X_j) > r - 2s$ then $\mathsf{Vol}((\widehat{\mathcal{R}}_i \cup \widehat{\mathcal{R}}_i') \cap (\widehat{\mathcal{R}}_j \cup \widehat{\mathcal{R}}_j')) \le (1-\epsilon)\pi r^2$,*
3) *if $s \ge r/7$ and $s = O(r)$ then $\mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}_j') \le (1-\epsilon)\pi r^2$,*
4) *if $s = \omega(r)$ then $\mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}_j') = O(r^3 \frac{s+1}{s}) = o(r^2)$.*

Proof:
(1) Assume w.l.o.g. that the segment $\overline{X_i X_j}$ is vertical and that X_i is above X_j. Let $\mathcal{S} \subset [0,1)^2$ be the upper halfcircle with center X_i and radius r, and $\widehat{\mathcal{S}} = \pi_1^{-1}(\mathcal{S}) = \mathcal{S} \times [0,1) \subset [0,1)^3$. Notice that $\mathsf{Vol}(\widehat{\mathcal{S}}) = \pi r^2/2$, $\widehat{\mathcal{S}} \subset \widehat{\mathcal{R}}_i$ and $\widehat{\mathcal{S}} \cap \widehat{\mathcal{R}}_j = \emptyset$, and the statement follows.

(2) The distance between X_i' and X_j' is greater than $3r/7$, since $d(X_i', X_j') \ge d(X_i, X_j) - 2s > r - 4s$. Let $\mathcal{S}_i$ ($\mathcal{S}_j$, respectively) be the set of points in $[0,1)^2$ at distance at most $8r/7$ from X_i' (X_j', respectively). Note that $\mathcal{S}_i$ and $\mathcal{S}_j$ are two circles of radius $8r/7$ with centers at distance greater than $3r/7$. Straightforward computations show that $\mathsf{Area}(\mathcal{S}_i \cap \mathcal{S}_j)$ is at most $(1-\epsilon)\pi r^2$ for some $\epsilon > 0$. We define $\widehat{\mathcal{S}}_i = \pi_1^{-1}(\mathcal{S}_i)$ and $\widehat{\mathcal{S}}_j = \pi_1^{-1}(\mathcal{S}_j)$. We have $\widehat{\mathcal{S}}_i \supset \widehat{\mathcal{R}}_i \cup \widehat{\mathcal{R}}_i'$ and $\widehat{\mathcal{S}}_j \supset \widehat{\mathcal{R}}_j \cup \widehat{\mathcal{R}}_j'$. Hence,

$$\begin{aligned}\mathsf{Vol}((\widehat{\mathcal{R}}_i \cup \widehat{\mathcal{R}}_i') \cap (\widehat{\mathcal{R}}_j \cup \widehat{\mathcal{R}}_j')) &\le \mathsf{Vol}(\widehat{\mathcal{S}}_i \cap \widehat{\mathcal{S}}_j) \\ &= \mathsf{Area}(\mathcal{S}_i \cap \mathcal{S}_j) \le (1-\epsilon)\pi r^2.\end{aligned}$$

(3) Let $k \in \{1,\dots,n\}$ be different from i and j. Observe that $\mathsf{Vol}(\widehat{\mathcal{R}}_i \setminus \widehat{\mathcal{R}}_j')$ is the probability that $d(X_i, X_k) \le r$ but $d(X_j', X_k') > r$. Suppose that $d(X_i, X_k) \le r$ but also $d(X_j', X_k) > 13r/14$, which happens with probability at least $(1 - 13^2/14^2)\pi r^2$. Let α be the angle of $\overrightarrow{X_j' X_k}$ with respect to the horizontal axis. Recall that vertex k moves between time steps t and $t+1$ towards a direction $2\pi z_k$, where z_k is the third coordinate of $\widehat{X}_k$. If $2\pi z_k \in [\alpha - \pi/3, \alpha + \pi/3]$, then the vertex increases its distance with respect to X_j' by at least $s/2 \ge r/14$, and thus $d(X_j', X_k') > r/14 + 13r/14 = r$. This range of directions has probability $1/3$. Summarizing, we proved that $\mathsf{Vol}(\widehat{\mathcal{R}}_i \setminus \widehat{\mathcal{R}}_j') \ge (1 - 13^2/14^2)\pi r^2/3$, and the statement follows.

(4) Given $k \in \{1,\dots,n\}$ different from i and j, observe that $\mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}_j')$ is the probability that $d(X_k, X_i) \le r$ and $d(X_k', X_j') \le r$. Suppose first that $s < 1/2$. We claim that the probability that $d(X_k', X_j') \le r$ conditional upon any fixed outcome of X_k is at most $(2+\epsilon)r/s$ for some $\epsilon > 0$, no matter which particular point X_k is chosen. In fact, assume $X_k \ne X_j'$ and let α be the angle of $\overrightarrow{X_k X_j'}$ with respect to the horizontal axis. If vertex k moves between steps t and $t+1$ towards a direction $2\pi z_k$ not in $[\alpha - \arcsin(r/s), \alpha + \arcsin(r/s)]$ then $d(X_k', X_j') > r$. Hence, $\mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}_j')$ is at most $\mathbf{P}(d(X_k, X_i) \le r) = \pi r^2$ times $(2+\epsilon)r/s$, which satisfies the claim. The case $X_k = X_j'$ is trivial.

The case $s \ge 1/2$ is similar, taking into consideration the fact that since vertex k may loop many times around the torus

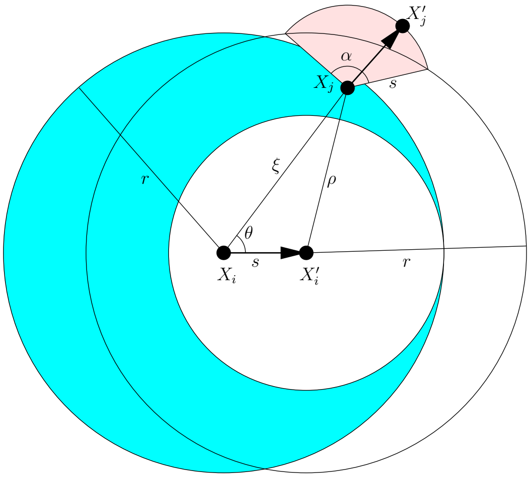

Fig. 1. Case 1 in the proof of Lemma 4.

while moving between steps t and $t+1$. In fact, as we move along the circumference of radius s centered at X_k, we cross the axes of the torus $\Theta(1+s)$ times. This gives the extra factor $(1+s)$ in the statement, which is negligible when $s = o(1)$ but grows large when $s = \omega(1)$. ■

For each $i \in \{1, \ldots, n\}$, we define $\widehat{\mathcal{Q}}_i = \widehat{\mathcal{R}}'_i \setminus \widehat{\mathcal{R}}_i$ and $\widehat{\mathcal{Q}}'_i = \widehat{\mathcal{R}}_i \setminus \widehat{\mathcal{R}}'_i$. Given any two vertices i and j, observe that $\widehat{X}_i \in \widehat{\mathcal{Q}}'_j$ iff $\widehat{X}_j \in \widehat{\mathcal{Q}}'_i$ iff $\mathsf{d}(X_i, X_j) \le r$ but $\mathsf{d}(X'_i, X'_j) > r$, i.e. the vertices are joined by an edge at time t but not at time $t+1$. This holds with probability $q = \mathsf{Vol}(\widehat{\mathcal{Q}}_i) = \mathsf{Vol}(\widehat{\mathcal{Q}}'_i)$, which neither depends on the particular vertices nor on t. The value of q depends on the asymptotic relation between r and s and is given in the following lemma.

Lemma 4. *The probability that two different vertices $i, j \in \{1, \ldots, n\}$ are at distance at most r at time t but greater than r at time $t+1$ is $q \le \pi r^2$, which also satisfies*

$$q \sim \begin{cases} \frac{4}{\pi} sr & \text{if } s = o(r), \\ \Theta(r^2) & \text{if } s = \Theta(r), \\ \pi r^2 & \text{if } s = \omega(r). \end{cases}$$

Proof: The first bound on q is immediate from the definition of q and the fact that $\mathsf{Vol}(\widehat{\mathcal{R}}_i) = \pi r^2$. In order to obtain the second statement, we consider three cases.

Case 1 ($s \le \epsilon r$, for some fixed but small enough $\epsilon > 0$). In order to compute the probability that $\widehat{X}_j \in \widehat{\mathcal{Q}}'_i$, we express $\widehat{X}_j = (x_j, y_j, z_j)$ in new coordinates (ρ, θ, z), where $\rho = d(X_j, X'_i)$, θ is the angle between the horizontal axis and $\overrightarrow{X_i X_j}$, and $z = z_j$. Integrate an element of volume over the region $\widehat{\mathcal{Q}}'_i$ in terms of these coordinates. Let $\xi = d(X_j, X_i)$, so that (ξ, θ, z) are the cylindrical coordinates (see Figure 1). Using the law of cosines, we write

$$\rho = \sqrt{\xi^2 + s^2 - 2\xi s \cos\theta} \quad \text{and} \qquad \qquad (4)$$
$$\xi = \sqrt{\rho^2 - s^2 \sin^2\theta} + s\cos\theta.$$

Notice that the minimum value that ρ can take is $r - s$, since X_j must lie outside the circle of radius $r - s$ and center X'_i. Otherwise as $d(X'_i, X'_j) \le r$, the vertices i and j would share an edge at step $t+1$. On the other hand, X_j must lie inside the circle of radius r centered at X_i, and setting $\xi = r$ in (4), we get that the maximum value ρ can achieve is $\sqrt{r^2 + s^2 - 2rs\cos\theta}$.

Let α be the angle determined from the range of all possible values of $2\pi z$ (i.e., possible directions for vertex j to move). By the law of cosines,

$$\alpha = 2\arccos\left(\frac{r^2 - s^2 - \rho^2}{2s\rho}\right).$$

From (4) and the change of variables formula, we can determine the element of volume in coordinates (ρ, θ, z):

$$dxdydz = \xi\, d\xi d\theta dz = \frac{\xi\rho}{\xi - s\cos\theta}\, d\rho d\theta dz.$$

Using the fact that $r - 2s \le \xi \le r$, we can write

$$\frac{\xi\rho}{\xi - s\cos\theta} = \rho\left(1 \pm O\left(\frac{s}{r}\right)\right).$$

Therefore,

$$\begin{aligned} q &= \int_{\widehat{\mathcal{Q}}'_i} dxdydz \\ &= \int_0^{2\pi} \int_{r-s}^{\sqrt{r^2+s^2-2rs\cos\theta}} \frac{\alpha}{2\pi} \frac{\xi\rho}{\xi - s\cos\theta}\, d\rho d\theta \\ &= \left(1 \pm O\left(\frac{s}{r}\right)\right) 2 \int_0^{\pi} \frac{1}{2\pi} \Big(- rs\sin\theta - \theta r^2 \\ &\quad + (r^2 + s^2 - 2rs\cos\theta) \arccos\frac{r\cos\theta - s}{\sqrt{r^2 + s^2 - 2rs\cos\theta}} \Big) d\theta. \end{aligned}$$

Looking at the Taylor series with respect to s/r of the expression inside the integral divided by r^2, we get

$$\begin{aligned} q &= \left(1 \pm O\left(\frac{s}{r}\right)\right) \int_0^{\pi} r^2 \left(-\frac{2\theta\cos\theta}{\pi}\frac{s}{r} + O\left(\left(\frac{s}{r}\right)^2\right)\right) d\theta \\ &= \left(1 \pm O\left(\frac{s}{r}\right)\right) \frac{4}{\pi} sr. \end{aligned}$$

Case 2 ($\epsilon r < s < r/7$). Recall that $\mathcal{R}_i$ is the circle of radius r and center X_i. Take the chord in $\mathcal{R}_i$ which is perpendicular to the segment $\overline{X_i X'_i}$ and at distance r from X'_i. This chord divides $\mathcal{R}_i$ into two regions. One of them, call it $\mathcal{S}$, has the property that all the points inside are at distance at least r from X'_i and moreover $\mathsf{Area}(\mathcal{S}) \ge \epsilon\sqrt{2\epsilon - \epsilon^2} r^2$. Suppose that $X_j \in \mathcal{S}$ (i.e., the vertex j is in $\mathcal{S}$ at time t), which happens with probability at least $\epsilon\sqrt{2\epsilon - \epsilon^2} r^2$. Let us now consider the circle centered at X'_i and passing through X_j. We observe that $d(X'_j, X'_i) > d(X_j, X'_i)$ with probability at least $1/2$, since it is sufficient that the direction $2\pi z_j$ in which vertex j moves lies in the outer side of the tangent of that circle at X_j. Therefore, the probability that $d(X_j, X_i) \le r$ and $d(X'_j, X'_i) > r$, or equivalently $\widehat{X}_j \in \widehat{\mathcal{Q}}'_i$, is at least $\frac{1}{2}\epsilon\sqrt{2\epsilon - \epsilon^2} r^2$.

Case 3 ($s \ge r/7$). We can write

$$q = \mathsf{Vol}(\widehat{\mathcal{Q}}'_i) = \mathsf{Vol}(\widehat{\mathcal{R}}_i \setminus \widehat{\mathcal{R}}'_i) = \mathsf{Vol}(\widehat{\mathcal{R}}_i) - \mathsf{Vol}(\widehat{\mathcal{R}}_i \cap \widehat{\mathcal{R}}'_i),$$

and the result follows from the statements (1) and (4) in Lemma 3.

■

We need the following technical result, which allows us to compute the probability that a given subset of $[0,1)^3$ contains no points of $\widehat{\mathcal{X}}$, but some other subsets contain at least one. Roughly speaking, the lemma shows that under some mild conditions the probability of having a certain number of points (including zero) in disjoint regions of the unit torus is asymptotically equal to the product of the probabilities of these events (that is, one can consider these events as if they were independent).

Lemma 5. *For any fixed integer $k \geq 0$, let $\widehat{\mathcal{S}}_0, \ldots, \widehat{\mathcal{S}}_k$ be pairwise disjoint subsets of $[0,1)^3$, with volumes $s_0, \ldots, s_k$ respectively. If $\sum_{i=0}^{k} s_i = o(1)$, then*

$$P = \mathbf{P}\left((\widehat{\mathcal{S}}_0 \cap \widehat{\mathcal{X}} = \emptyset) \wedge \bigwedge_{i=1}^{k} (\widehat{\mathcal{S}}_i \cap \widehat{\mathcal{X}} \neq \emptyset) \right) \sim (1-s_0)^n \prod_{i=1}^{k} \left(1 - e^{-s_i n}\right).$$

Proof: Using inclusion-exclusion,

$$P = \sum_{c_j \in \{0,1\},\ 2 \leq j \leq i} (-1)^{\sum_{j=2}^{i} c_j} \left(1 - \left(s_1 + \sum_{j=2}^{i} c_j s_j\right)\right)^n,$$

Let $\alpha = \left(1 - \left(s_1 + \sum_{j=2}^{i-1} c_j s_j\right)\right)^n$. Then,

$$P \sim \left(1 - e^{-s_i n}\right) \sum_{c_j \in \{0,1\},\ 2 \leq j \leq i-1} (-1)^{\sum_{j=2}^{i-1} c_j} \times \alpha,$$

and the argument follows by induction. ∎

Next, we study the changes of isolated vertices between two consecutive steps t and $t+1$. Let $K_{1,t}$ be the number of isolated vertices in $G(\mathcal{X}_t; r)$. For any two consecutive steps t and $t+1$, define the following random variables:

- B_t is the number of vertices i such that X_i is not isolated in $G(\mathcal{X}_t; r)$ but X'_i is isolated in $G(\mathcal{X}_{t+1}; r)$;
- D_t is the number of vertices i such that X_i is isolated in $G(\mathcal{X}_t; r)$ but X'_i is not isolated in $G(\mathcal{X}_{t+1}; r)$;
- S_t is the number of vertices i such that X_i and X'_i are both isolated in $G(\mathcal{X}_t; r)$ and $G(\mathcal{X}_{t+1}; r)$.

Denote them by B, D and S whenever t and $t+1$ are understood. Note that B and D have the same distribution.

Recall that given a collection of events $\mathcal{E}_1(n), \ldots, \mathcal{E}_k(n)$ and of random variables $W_1(n), \ldots, W_l(n)$ taking values in $\mathbb{N}$, with k and l fixed, they are defined to be *mutually asymptotically independent* if for any $k', l', i_1, \ldots, i_{k'} \in \mathbb{N}$ and $j_1, \ldots, j_{l'}, w_1, \ldots, w_{l'} \in \mathbb{N}$ such that $k' \leq k$, $l' \leq l$, $1 \leq i_1 < \cdots < i_{k'} \leq k$, $1 \leq j_1 < \cdots < j_{l'} \leq l$, we have

$$\mathbf{Pr}\left[\bigwedge_{a=1}^{k'} \mathcal{E}_{i_a} \wedge \bigwedge_{b=1}^{l'} (W_{j_b} = w_b) \right] \sim \prod_{a=1}^{k'} \mathbf{Pr}\left[\mathcal{E}_{i_a}\right] \prod_{b=1}^{l'} \mathbf{Pr}\left[W_{j_b} = w_b\right]. \tag{5}$$

The next Proposition, characterizes the changes of the number of isolated vertices between two consecutive steps. The proof is based on the computation of the joint factorial moments of the variables accounting for these changes. At first sight, it is not obvious that the probability of existence of components of larger sizes is negligible compared to the probability of sudden appearance of isolated vertices, but this is indeed shown in Lemma 7.

Proposition 6. *Assume $\mu = \Theta(1)$. Then for any two consecutive steps,*

$$\mathbf{E}B = \mathbf{E}D \sim \mu(1 - e^{-qn}) \quad \textit{and} \quad \mathbf{E}S \sim \mu e^{-qn}.$$

Moreover we have that

- *If $s = o(1/rn)$, then $\mathbf{P}(B > 0) \sim \mathbf{E}B$; $\mathbf{P}(D > 0) \sim \mathbf{E}D$; S is asymptotically Poisson; and $(B > 0)$, $(D > 0)$ and S are asymptotically mutually independent.*
- *If $s = \Theta(1/rn)$, then B, D and S are asymptotically mutually independent Poisson.*
- *If $s = \omega(1/rn)$, then B and D are asymptotically Poisson; $\mathbf{P}(S > 0) \sim \mathbf{E}S$; and B, D and $(S > 0)$ are asymptotically mutually independent.*

Proof: The central ingredient in the proof is the computation of the joint factorial moments $\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3})$ of these variables. In particular, we find the asymptotic values of $\mathbf{E}B$, $\mathbf{E}D$ and $\mathbf{E}S$. Moreover, in the case $s = \Theta\big(1/(rn)\big)$, we show that for any fixed naturals ℓ_1, ℓ_2 and ℓ_3 we have

$$\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3}) \sim (\mathbf{E}B)^{\ell_1}(\mathbf{E}D)^{\ell_2}(\mathbf{E}S)^{\ell_3}. \tag{6}$$

The statement then follows from Theorem 1.23 in [3].

The other cases are more delicate since (6) does not always hold for extreme values of s, and we obtain a weaker result. In the case $s = o\big(1/(rn)\big)$, we compute the moments for any natural ℓ_3 but only for $\ell_1, \ell_2 \in \{0,1,2\}$ and obtain

$$\begin{aligned} &\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3}) \sim (\mathbf{E}B)^{\ell_1}(\mathbf{E}D)^{\ell_2}(\mathbf{E}S)^{\ell_3}, \quad \text{if } \ell_1, \ell_2 < 2, \\ &\mathbf{E}([B]_2[D]_{\ell_2}[S]_{\ell_3}) = o(\mathbf{E}(B\,[D]_{\ell_2}[S]_{\ell_3})), \\ &\mathbf{E}([B]_{\ell_1}[D]_2[S]_{\ell_3}) = o(\mathbf{E}([B]_{\ell_1}D\,[S]_{\ell_3})). \end{aligned} \tag{7}$$

Using the upper and lower bounds in [3], Section 1.4, we get that $(B > 0)$, $(D > 0)$ and S satisfy (5), and $\mathbf{P}(B > 0) \sim \mathbf{E}B$, $\mathbf{P}(D > 0) \sim \mathbf{E}D$ and $\forall k \in \mathbb{N}$, $\mathbf{P}(S = k) \sim e^{-\mathbf{E}S}\frac{(\mathbf{E}S)^k}{k!}$. For the case $s = \omega\big(1/(rn)\big)$, we compute the moments for any naturals ℓ_1 and ℓ_2 but only for $\ell_3 \in \{0,1,2\}$ and obtain

$$\begin{aligned} &\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3}) \sim (\mathbf{E}B)^{\ell_1}(\mathbf{E}D)^{\ell_2}(\mathbf{E}S)^{\ell_3}, \quad \text{if } \ell_3 < 2, \\ &\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_2) = o(\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}S)) \end{aligned} \tag{8}$$

From this and by using once more upper and lower bounds given in Section 1.4 of [3], we conclude that B, D and $(S > 0)$ satisfy (5), and

$$\mathbf{P}(B = k) \sim e^{-\mathbf{E}B}\frac{(\mathbf{E}B)^k}{k!} \quad \forall k \in \mathbb{N},$$

$$\mathbf{P}(D = k) \sim e^{-\mathbf{E}D}\frac{(\mathbf{E}D)^k}{k!} \quad \forall k \in \mathbb{N} \text{ and } \mathbf{P}(S > 0) \sim \mathbf{E}S.$$

We proceed to compute the moments. For each $i \in \{1, \ldots, n\}$, define B_i, D_i and S_i as the indicator functions of the following events: $B_i = 1$ iff X_i is not isolated in $G(\mathcal{X}_t; r)$, but X'_i is isolated in $G(\mathcal{X}_{t+1}; r)$; $D_i = 1$ iff X_i is isolated in $G(\mathcal{X}_t; r)$

but X_i' is not isolated in $G(\mathcal{X}_{t+1};r)$, and $S_i=1$ iff X_i and X_i' are both isolated in $G(\mathcal{X}_t;r)$ and $G(\mathcal{X}_{t+1};r)$. Then,

$$B=\sum_{i=1}^n B_i,\quad D=\sum_{i=1}^n D_i,\quad S=\sum_{i=1}^n S_i.$$

Recall that $\widehat{\mathcal{Q}}_i=\widehat{\mathcal{R}}_i'\setminus\widehat{\mathcal{R}}_i$ and $\widehat{\mathcal{Q}}_i'=\widehat{\mathcal{R}}_i\setminus\widehat{\mathcal{R}}_i'$. Note that $B_i=1$ iff all points in $\widehat{\mathcal{X}}\setminus\{\widehat{X}_i\}$ are outside $\widehat{\mathcal{R}}_i'$ but at least one is inside $\widehat{\mathcal{Q}}_i'$; $D_i=1$ iff all points in $\widehat{\mathcal{X}}\setminus\{\widehat{X}_i\}$ are outside $\widehat{\mathcal{R}}_i$ but at least one is inside $\widehat{\mathcal{Q}}_i$; and finally $S_i=1$ iff all points in $\widehat{\mathcal{X}}\setminus\{\widehat{X}_i\}$ are outside $\widehat{\mathcal{R}}_i\cup\widehat{\mathcal{R}}_i'=\widehat{\mathcal{R}}_i\cup\widehat{\mathcal{Q}}_i=\widehat{\mathcal{R}}_i'\cup\widehat{\mathcal{Q}}_i'$.

Given any fixed naturals ℓ_1,ℓ_2,ℓ_3 with $\ell=\ell_1+\ell_2+\ell_3$, we choose an ordered tuple J of ℓ different vertices $i_1,\dots,i_\ell\in\{1,\dots,n\}$, and define

$$\mathcal{E}=\bigwedge_{a=1}^{\ell_1}(B_{i_a}=1)\wedge\bigwedge_{b=\ell_1+1}^{\ell_1+\ell_2}(D_{i_b}=1)\wedge\bigwedge_{c=\ell_1+\ell_2+1}^{\ell}(S_{i_c}=1).\tag{9}$$

Then $\mathbf{P}(\mathcal{E})$ does not depend on the particular tuple J, and multiplying it by the number $[n]_\ell$ of ordered choices of J, we get

$$\mathbf{E}([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3})=[n]_\ell\mathbf{P}(\mathcal{E}).\tag{10}$$

Relabelling the vertices as $J=(1,\dots,\ell)$, let $\widehat{\mathcal{Y}}=\bigcup_{i=1}^{\ell}\{\widehat{X}_i\}$. Define the set

$$\widehat{\mathcal{R}}=\bigcup_{i=1}^{\ell_1}\widehat{\mathcal{R}}_i'\cup\bigcup_{i=\ell_1+1}^{\ell_1+\ell_2}\widehat{\mathcal{R}}_i\cup\bigcup_{i=\ell_1+\ell_2+1}^{\ell}(\widehat{\mathcal{R}}_i\cup\widehat{\mathcal{R}}_i'),$$

and the collection of sets

$$\widehat{\boldsymbol{\mathcal{Q}}}=\{\widehat{\mathcal{Q}}_1',\dots,\widehat{\mathcal{Q}}_{\ell_1}',\widehat{\mathcal{Q}}_{\ell_1+1},\dots,\widehat{\mathcal{Q}}_{\ell_1+\ell_2}\}.$$

Rename $\widehat{\mathcal{Q}}_i^*=\widehat{\mathcal{Q}}_i'$ for $1\le i\le\ell_1$, $\widehat{\mathcal{Q}}_i^*=\widehat{\mathcal{Q}}_i$ for $\ell_1+1\le i\le\ell_1+\ell_2$, so $\widehat{\boldsymbol{\mathcal{Q}}}=\{\widehat{\mathcal{Q}}_1^*,\dots,\widehat{\mathcal{Q}}_{\ell_1+\ell_2}^*\}$.

Case 1 ($s=\Theta(1/rn)$). Say that a vertex $i\in J$ is *restricted* if there is some other $j\in J$ with $j>i$ such that $d(X_i,X_j)\le 2r+4s$. Let $\mathcal{F}$ be the event that $d(X_i,X_j)>2r+4s$ for all $i,j\in J$ $(i\neq j)$. This event has probability $1-O(r^2)$. Assume first that $\mathcal{F}$ holds and compute the probability of $\mathcal{E}$ conditional upon that. Notice that $\mathcal{F}$ implies that for any $i,j\in J$ $(i\neq j)$ we must have $\widehat{\mathcal{R}}_i\cap\widehat{\mathcal{R}}_j=\emptyset$, $\widehat{\mathcal{R}}_i'\cap\widehat{\mathcal{R}}_j'=\emptyset$ and $\widehat{\mathcal{R}}_i\cap\widehat{\mathcal{R}}_j'=\emptyset$. Then $\mathsf{Vol}(\widehat{\mathcal{R}})=\ell\pi r^2+\ell_3 q$, and the sets in $\widehat{\boldsymbol{\mathcal{Q}}}$ are pairwise disjoint and also disjoint from $\widehat{\mathcal{R}}$. Moreover observe that, conditional upon $\mathcal{F}$, $\mathcal{E}$ is equivalent to the event that all points in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$ lie outside $\widehat{\mathcal{R}}$, but at least one belongs to each $\widehat{\mathcal{Q}}_i^*\in\widehat{\boldsymbol{\mathcal{Q}}}$. From Lemmata 4 and 5, we get:

$$\begin{aligned}\mathbf{P}(\mathcal{E}\wedge\mathcal{F})&=(1-O(r^2))\,\mathbf{P}(\mathcal{E}\mid\mathcal{F})\\&\sim\left(\frac{\mu}{n}\right)^{\ell}(1-e^{-qn})^{\ell_1+\ell_2}e^{-\ell_3 qn}.\end{aligned}\tag{11}$$

We claim that $\mathbf{P}(\mathcal{E}\wedge\mathcal{F})$ is the main contribution to $\mathbf{P}(\mathcal{E})$. In fact, if $\mathcal{F}$ does not hold then $\mathbf{P}(\mathcal{E}\mid\overline{\mathcal{F}})$ is larger than the expression in (11), but this is balanced out by the fact that $\mathbf{P}(\overline{\mathcal{F}})$ is small. Before proving this claim, define $\mathcal{H}$ to be the event that $d(X_i,X_j)>r-2s$ for all $i,j\in J$ $(i\neq j)$. Notice that $\mathcal{E}$ implies $\mathcal{H}$, since otherwise, for some $i,j\in J$, X_i and X_j would be joined by an edge in $G(\mathcal{X}_t;r)$, and also X_i' and X_j' in $G(\mathcal{X}_{t+1};r)$, which is not compatible with $\mathcal{E}$. Therefore we only need to prove that $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}})=\mathbf{P}(\overline{\mathcal{F}}\wedge\mathcal{H})\mathbf{P}(\mathcal{E}\mid\overline{\mathcal{F}}\wedge\mathcal{H})$ is negligible compared to (11).

Suppose then that $\mathcal{H}$ holds and that a number $p>0$ of vertices in J are *restricted*, i.e. $\mathcal{F}$ does not hold. This happens with probability $O(r^{2p})$. In this case, as each unrestricted vertex in J contributes at least with πr^2 to $\mathsf{Vol}(\widehat{\mathcal{R}})$ and the first restricted one gives by Lemma 3 (2) the term $\epsilon\pi r^2$, we get $\mathsf{Vol}(\widehat{\mathcal{R}})\ge(\ell-p)\pi r^2+\epsilon\pi r^2$. Moreover, $\mathcal{E}$ implies that all points in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$ lie outside of $\widehat{\mathcal{R}}$, which has probability $\big(1-\mathsf{Vol}(\widehat{\mathcal{R}})\big)^{n-\ell}=O(1/n^{\ell-p+\epsilon})$. Summarizing, the weight in $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}})$ coming from situations with p restricted vertices is $O(r^{2p}/n^{\ell-p+\epsilon})=O(\log^p n/n^{\ell+\epsilon})$, and is thus negligible compared to (11). Hence $\mathbf{P}(\mathcal{E})\sim\mathbf{P}(\mathcal{E}\wedge\mathcal{F})$, and the required condition on the moments announced in (6) follows from (10) and (11).

Case 2 ($s=o(1/rn)$). Defining $\mathcal{F}$ and $\mathcal{H}$ as in the case $s=\Theta\big(1/(rn)\big)$ and by an analogous argument, we obtain

$$\mathbf{P}(\mathcal{E}\wedge\mathcal{F})\sim\left(\frac{\mu}{n}\right)^{\ell}(1-e^{-qn})^{\ell_1+\ell_2}e^{-\ell_3 qn}\sim\left(\frac{\mu}{n}\right)^{\ell}(qn)^{\ell_1+\ell_2}.\tag{12}$$

However, the analysis of the case that $\mathcal{F}$ does not hold is slightly more delicate here. Indeed, there is an additional $o(1)$ factor in (12), namely $(qn)^{\ell_1+\ell_2}$, which forces us to get tighter bounds on $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}}\wedge\mathcal{H})$ than the ones obtained before. Unlike in the case $s=\Theta\big(1/(rn)\big)$, we need to consider the role of $\widehat{\boldsymbol{\mathcal{Q}}}$ when $\mathcal{F}$ does not hold, and special care must be taken with several new situations which do not occur otherwise. For instance, since the elements of $\widehat{\boldsymbol{\mathcal{Q}}}$ are not necessarily disjoint, then for $\widehat{\mathcal{Q}}_i^*,\widehat{\mathcal{Q}}_j^*\in\widehat{\boldsymbol{\mathcal{Q}}}$ the condition that both contain some element of $\widehat{\mathcal{X}}$ can be satisfied by having just a single point in $\widehat{\mathcal{Q}}_i^*\cap\widehat{\mathcal{Q}}_j^*\cap\widehat{\mathcal{X}}$. Moreover, if $\ell_1\ge2$ and $1\le i,j\le\ell_1$ (or $\ell_2\ge2$ and $\ell_1+1\le i,j\le\ell_1+\ell_2$), the previous condition is also satisfied if $\widehat{X}_j\in\widehat{\mathcal{Q}}_i^*$, which is equivalent to $\widehat{X}_i\in\widehat{\mathcal{Q}}_j^*$. If the latter situation occurs, we say that i and j *collaborate*.

We bound the weight in $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}})$ due to situations in which there are no pairs of elements in J which collaborate. Let $J_1=\{1,\dots,\ell_1+\ell_2\}$ and $\widehat{\mathcal{Y}}_1=\bigcup_{i=1}^{\ell_1+\ell_2}\{\widehat{X}_i\}$, and consider the set $\mathscr{P}$ of partitions of J_1. A partition of J_1 is a collection of nonempty subsets of J_1, denoted *blocks*, which are disjoint and have union J_1. The size of a partition is the number of blocks, and for each block we call *leader* to the maximal element in the block. Given a partition $P=\{A_1,\dots,A_k\}\in\mathscr{P}$ and given $i_1,\dots,i_k\in\{1,\dots,n\}\setminus J$, let $\mathcal{E}_{P,i_1,\dots,i_k}$ be the following event: For each block A_j of P, we have $\widehat{X}_{i_j}\in\bigcap_{i\in A_j}\widehat{\mathcal{Q}}_i^*$ and moreover, all the points in $\widehat{\mathcal{X}}\setminus(\widehat{\mathcal{Y}}\cup\{i_1,\dots,i_k\})$ lie outside of $\widehat{\mathcal{R}}$. We wish to bound the probability of $\mathcal{E}_{P,i_1,\dots,i_k}\wedge\overline{\mathcal{F}}\wedge\mathcal{H}$. Notice that if $\mathcal{E}_{P,i_1,\dots,i_k}$ holds, then all the $\ell_1+\ell_2-k$ non-leader elements in J_1 must be restricted, and possibly some other p' vertices in J are restricted too. Moreover, $\mathcal{F}$ does not hold iff this p' satisfies $0<\ell_1+\ell_2-k+p'<\ell$. Given any p' with that property, suppose that p' is exactly the number of restricted vertices in J which are either in $J\setminus J_1$ or are leaders of some block. We condition upon this and also upon $\mathcal{H}$, which has probability $r^{2p'}$. Then for each block A_j with leader l_j, event $\mathcal{E}_{P,i_1,\dots,i_k}$ requires that $\widehat{X}_{i_j}\in\widehat{\mathcal{Q}}_{l_j}^*$ and for all $i\in A_j$ $(i\neq l_j)$, $\widehat{X}_i\in(\widehat{\mathcal{Q}}_{i_j}\cup\widehat{\mathcal{Q}}_{i_j}')$. In addition, since the number of restricted vertices in J is

$\ell_1+\ell_2-k+p'>0$, arguing as in the case $s=\Theta\big(1/(rn)\big)$, we have $\mathsf{Vol}(\widehat{\mathcal{R}})\geq(\ell_3+k-p')\pi r^2+\epsilon\pi r^2$. The contribution to $\mathbf{P}(\mathcal{E}_{P,i_1,\dots,i_k}\wedge\overline{\mathcal{F}}\wedge\mathcal{H})$ for this particular p' is

$$\begin{aligned}O(r^{2p'})q^k(2q)^{\ell_1+\ell_2-k}(1-\mathsf{Vol}(\widehat{\mathcal{R}}))^{n-\ell-k}\\=O\left(\frac{\log^{p'}n}{n^{\ell+k+\epsilon}}\right)(qn)^{\ell_1+\ell_2},\end{aligned}$$

therefore for some $0<\epsilon'<\epsilon$, we can write

$$\mathbf{P}(\mathcal{E}_{P,i_1,\dots,i_k}\wedge\overline{\mathcal{F}}\wedge\mathcal{H})=O\left(\frac{1}{n^{\ell+k+\epsilon'}}\right)(qn)^{\ell_1+\ell_2}.$$

Finally, observe that if there are no pairs of elements in J which collaborate, then $\mathcal{E}\wedge\overline{\mathcal{F}}$ implies that $\mathcal{E}_{P,i_1,\dots,i_k}\wedge\overline{\mathcal{F}}\wedge\mathcal{H}$ holds for some $P\in\mathscr{P}$ of size k and some $i_1,\dots,i_k\in\{1,\dots,n\}\setminus J$, and therefore has probability

$$O\left(n^k\right)O\left(\frac{1}{n^{\ell+k+\epsilon'}}\right)(qn)^{\ell_1+\ell_2}=O\left(\frac{1}{n^{\ell+\epsilon'}}\right)(qn)^{\ell_1+\ell_2},$$

which is negligible compared to (12). In particular, if $\ell_1,\ell_2<2$, then no pair of elements in J collaborates and then $\mathbf{P}(\mathcal{E})\sim\mathbf{P}(\mathcal{E}\wedge\mathcal{F})$. Hence, the first line of (7) follows from (10) and (12).

We extend the approach above to deal with situations in which some pair of elements in J collaborate. Notice that if $s\to 0$ fast, their contribution to $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}}\wedge\mathcal{H})$ may be larger than (12). Hence we restrict ℓ_1 and ℓ_2 to be at most 2 and prove only (7). If $\ell_1=2$, let $\mathcal{E}_1$ be the following event: $\widehat{X}_1\in\widehat{\mathcal{Q}}'_2$; $\widehat{\mathcal{R}}$ contains no points in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$; and for each natural i, $3\leq i\leq 2+\ell_2$, $\widehat{\mathcal{Q}}_i$ contains some point in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$. If $\ell_2=2$, let $\mathcal{E}_2$ be the following event: $\widehat{X}_{\ell_1+1}\in\widehat{\mathcal{Q}}_{\ell_1+2}$; $\widehat{\mathcal{R}}$ contains no points in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$; and for each natural i, $1\leq i\leq\ell_1$, $\widehat{\mathcal{Q}}'_i$ contains some point in $\widehat{\mathcal{X}}\setminus\widehat{\mathcal{Y}}$. Finally if $\ell_1=\ell_2=2$, let $\mathcal{E}_{1,2}$ be the following event: $\widehat{X}_1\in\widehat{\mathcal{Q}}'_2$ and $\widehat{X}_3\in\widehat{\mathcal{Q}}_4$.

In order to compute $\mathbf{P}(\mathcal{E}_1\wedge\mathcal{H})$, we can repeat the same argument as above, but imposing that $\widehat{X}_1\in\widehat{\mathcal{Q}}'_2$ and ignoring other conditions on $\widehat{\mathcal{Q}}'_1$ and $\widehat{\mathcal{Q}}'_2$. We get that for some $\epsilon'>0$

$$\mathbf{P}(\mathcal{E}_1\wedge\mathcal{H})=O\left(\frac{1}{n^{\ell-1+\epsilon'}}\right)q(qn)^{\ell_2}=O\left(\frac{1}{n^{\ell+\epsilon'}}\right)(qn)^{1+\ell_2},\tag{13}$$

and similarly

$$\begin{aligned}\mathbf{P}(\mathcal{E}_2\wedge\mathcal{H})&=O\left(\frac{1}{n^{\ell+\epsilon'}}\right)(qn)^{\ell_1+1},\\\mathbf{P}(\mathcal{E}_{1,2}\wedge\mathcal{H})&=O\left(\frac{1}{n^{\ell+\epsilon'}}\right)(qn)^2.\end{aligned}\tag{14}$$

Observe that if some vertices in J collaborate, then $\mathcal{E}\wedge\overline{\mathcal{F}}$ implies that $\mathcal{E}_1\wedge\mathcal{H}$, $\mathcal{E}_2\wedge\mathcal{H}$ or $\mathcal{E}_{1,2}\wedge\mathcal{H}$ hold. Unfortunately, from (12), (13) and (14) we cannot guarantee that $\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}})$ is smaller than $\mathbf{P}(\mathcal{E}\wedge\mathcal{F})$, but by (10), multiplying these probabilities by $[n]_\ell$ we get (7).

Case 3 ($s=\omega(1/rn)$, but also $s=O(r)$). Following the same notation as in the case $s=\Theta(1/rn)$ and by an analogous argument, we obtain

$$\mathbf{P}(\mathcal{E}\wedge\mathcal{F})\sim\left(\frac{\mu}{n}\right)^\ell(1-e^{-qn})^{\ell_1+\ell_2}e^{-\ell_3qn}\sim\left(\frac{\mu}{n}\right)^\ell e^{-\ell_3qn}.\tag{15}$$

If $\ell_3\leq 1$, we claim that (15) is the main contribution to $\mathbf{P}(\mathcal{E})$. In fact, suppose that $\mathcal{H}$ holds and that $p>0$ of the vertices in J are restricted ($\mathcal{F}$ does not hold), which happens with probability $O(r^{2p})$. Since $\ell_3\leq 1$, then the only possible event which contributes to S in the definition of $\mathcal{E}$ is $(S_\ell=1)$ (cf. (9)). This involves vertex ℓ, which cannot be restricted by definition. Therefore $\mathsf{Vol}(\widehat{\mathcal{R}})\geq(\ell-p)\pi r^2+\ell_3q+\epsilon\pi r^2$, since by (2) and (3) in Lemma 3, the unrestricted vertices in J contribute $(\ell-p)\pi r^2+\ell_3q$ to $\mathsf{Vol}(\widehat{\mathcal{R}})$, and the first restricted one gives the term $\epsilon\pi r^2$. Therefore in this situation, the probability of $\mathcal{E}$ is $O(e^{-\ell_3qn}/n^{\ell-p+\epsilon})$, which combined with the probability $O(r^{2p})$, that p vertices are restricted, has negligible weight compared to (15). Hence, $\mathbf{P}(\mathcal{E})\sim\mathbf{P}(\mathcal{E}\wedge\mathcal{F})$, and the first line of (8) follows from (10) and (15).

If $\ell_3=2$ and we have p restricted vertices in J, we can only assure that $\mathsf{Vol}(\widehat{\mathcal{R}})\geq(\ell-p)\pi r^2+q+\epsilon\pi r^2$. Then for some $0<\epsilon'<\epsilon$,

$$\mathbf{P}(\mathcal{E}\wedge\overline{\mathcal{F}})=O\left(\frac{r^{2p}}{n^{\ell-p+\epsilon}}\right)e^{-qn}=O\left(\frac{1}{n^{\ell+\epsilon'}}\right)e^{-qn}.\tag{16}$$

Using (10), (15) and (16), we verify that the second line of (8) is satisfied.

Case 4 ($s=\omega(r)$). Let $\mathcal{F}'$ be the event that for any $i,j\in J$ ($i\neq j$) we have that $d(X_i,X_j)>2r$ and $d(X'_i,X'_j)>2r$. This event has probability $1-O(r^2)$. Observe that if $\mathcal{F}'$ holds, then for any $i,j\in J$ ($i\neq j$) we must have $\widehat{\mathcal{R}}_i\cap\widehat{\mathcal{R}}_j=\emptyset$, $\widehat{\mathcal{R}}'_i\cap\widehat{\mathcal{R}}'_j=\emptyset$ and $\widehat{\mathcal{R}}_i\cap\widehat{\mathcal{R}}'_j=\emptyset$. Therefore, $\mathsf{Vol}(\widehat{\mathcal{R}})=\ell\pi r^2+\ell_3q$ and the sets in $\widehat{\boldsymbol{\mathcal{Q}}}$ are pairwise disjoint and also disjoint from $\widehat{\mathcal{R}}$. Using Lemmata 4 and 5, and by the same argument that leads to (11),

$$\mathbf{P}(\mathcal{E}\wedge\mathcal{F}')\sim\left(\frac{\mu}{n}\right)^\ell(1-e^{-qn})^{\ell_1+\ell_2}e^{-\ell_3qn}\sim\left(\frac{\mu}{n}\right)^\ell e^{-\ell_3qn}.$$

The remaining of the argument is analogous to Case 3 but replacing $\mathcal{F}$ with $\mathcal{F}'$ and using Lemma 3 (3).

■

Taking into account that $K_{1,t}=D_t+S_t$ and $K_{1,t+1}=S_t+B_t$, Proposition 6 completely characterizes the number of isolated vertices at two consecutive steps in the case $s=\Theta\big(1/(rn)\big)$. For the other ranges of s, the result is weaker but still sufficient for our further purposes. We remark that if $s=o\big(1/(rn)\big)$, then creations and destructions of isolated vertices are rare, but a Poisson number of isolated vertices is present at both consecutive steps. Otherwise if $s=\omega\big(1/(rn)\big)$, then the isolated vertices which are present at both consecutive steps are rare since, but a Poisson number of them are created and also a Poisson number destroyed.

To characterize the connectivity of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$, we need to bound the probability of the event that components other than isolated vertices and the giant one appear at some step. Recall that in the static case, a.a.s. this does not occur at one single step t [7]. However, during long periods of time this event could affect the connectivity and must be considered.

Given a component Γ of $G(\mathcal{X};r)$, Γ is *embeddable* if it can be mapped into the square $[r,1-r]^2$ by a translation in the torus. Embeddable components do not wrap around the torus. Components which are not embeddable must have a size of at least $\Omega(1/r)$ (see Figure 2).

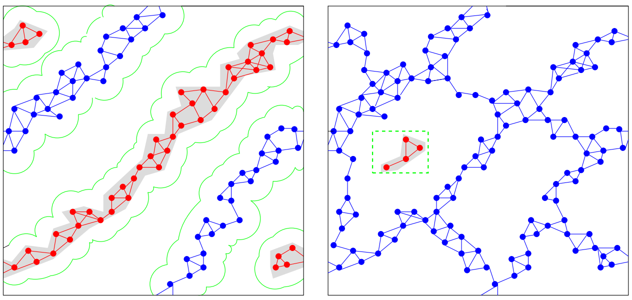

Fig. 2. *Left picture:* Two non-embeddable components which are not solitary. *Right picture:* One solitary component and one embeddable component (shaded).

Sometimes several non-embeddable components can coexist together. However, there are some non-embeddable components which are so spread around the torus, that they do not allow any room for other non-embeddable ones. Call these components *solitary* (see Figure 2). By definition, we can have at most one solitary component. We cannot disprove the existence of a solitary component, since with probability $1-o(1)$ there exists a giant component of this nature. For components which are not solitary, we give asymptotic bounds on the probability of their existence according to their size.

The proof of the next lemma is an extension of the proofs of Lemma 4 and Lemma 5 in [7], where exact probabilities for the existence of components of size $\ell \geq 2$ are computed for the static model $G(\mathcal{X};r)$. In the setup of the current paper, new difficulties arise, since we must also take into consideration changes between two consecutive time steps. The basic idea is that at a step t, if $\widetilde{K}_{2,t}$ denotes the number of non-solitary components other than isolated vertices occurring at t, we show that in the dynamic evolution of connectivity, those components have a negligible effect when compared to the isolated vertices.

Lemma 7. *Assume that* $\mu = \Theta(1)$ *and* $s = o\big(1/(rn)\big)$*. Then,*

- $\mathbf{P}(\widetilde{K}_{2,t} > 0 \wedge \widetilde{K}_{2,t+1} = 0) = \mathbf{P}(\widetilde{K}_{2,t} = 0 \wedge \widetilde{K}_{2,t+1} > 0) = o(srn)$,
- $\mathbf{P}(\widetilde{K}_{2,t} > 0 \wedge B_t > 0) = o(srn)$.

Proof: Recall from Lemma 4 that if $s = o\big(1/(rn)\big)$ then $q = \Theta(rs)$. It suffices to prove that $\mathbf{P}(\widetilde{K}_{2,t} > 0 \wedge \widetilde{K}_{2,t+1} = 0) = o(qn)$ and $\mathbf{P}(\widetilde{K}_{2,t} > 0 \wedge B_t > 0) = o(qn)$, since $(\widetilde{K}_{2,t} = 0 \wedge \widetilde{K}_{2,t+1} > 0)$ corresponds in the time-reversed process to $(\widetilde{K}_{2,t} > 0 \wedge \widetilde{K}_{2,t+1} = 0)$, and thus they have the same probability.

Consider all the possible components in $G(\mathcal{X};r)$ which are not solitary and have size at least 2. They are classified into several types according to their size and diameter, and we deal with each type separately. Denote by M_i the number of components of *type* i in $G(\mathcal{X}_t;r)$, we must show that for each type i

$$\begin{aligned} &\mathbf{P}(M_i > 0 \wedge \widetilde{K}_{2,t+1} = 0) = o(qn) \quad \text{and} \\ &\mathbf{P}(M_i > 0 \wedge B_t > 0) = o(qn). \end{aligned} \tag{17}$$

The following definition describes the changes of edges between $G(\mathcal{X}_t;r)$ and $G(\mathcal{X}_{t+1};r)$. For each $i \in \{1,\dots,n\}$ we define $\widehat{\mathcal{P}}_i = \widehat{\mathcal{Q}}_i \cup \widehat{\mathcal{Q}}'_i = \widehat{\mathcal{R}}_i \Delta \widehat{\mathcal{R}}'_i$ (where Δ denotes the symmetric difference of sets). Given also $j \in \{1,\dots,n\}$, observe that $\widehat{X}_j \in \widehat{\mathcal{P}}_i$ iff $\widehat{X}_i \in \widehat{\mathcal{P}}_j$ iff vertices i and j share an edge either at time t or at time $t+1$ but not at both times, which happens with probability $\mathsf{Vol}(\widehat{\mathcal{P}}_i) = 2q$. Throughout this proof let $\epsilon = 10^{-18}$.

Part 1. First we consider all possible embeddable components in $G(\mathcal{X};r)$ with diameter between ϵr and $6\sqrt{2}r$. Call them components of *type* 1, and let M_1 denote their number at time t.

The argument of this part follows the lines to the proof of Part 3 in Lemma 5 of [7], but taking into consideration the peculiarities of the fact that the graph is dynamic. We tessellate the torus $[0,1)^2$ into square cells of side αr, for some fixed but small enough $\alpha > 0$. Let Γ be a component of *type* 1, and let $\mathcal{S} = \mathcal{S}_\Gamma$ be the set of all points in the torus $[0,1)^2$ which are at distance at most r from some vertex in Γ. Remove from $\mathcal{S}$ the vertices of Γ and the edges (represented by straight line segments) and denote by $\mathcal{S}'$ the outer connected topological component of the remaining set. By construction, $\mathcal{S}'$ must contain no vertex in $\mathcal{X}$ (see Figure 3, left picture).

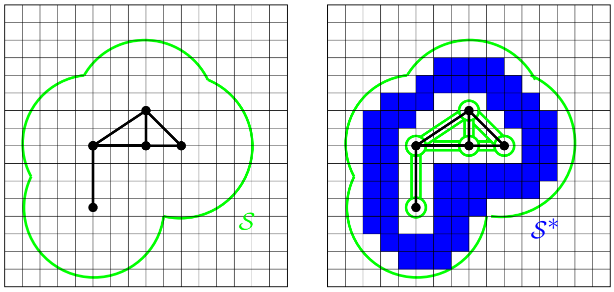

Fig. 3. The tessellation for counting components of *type* 1

Now let i_{L}, i_{R}, i_{T} and i_{B}, respectively, be the indices of the leftmost, rightmost, topmost and bottommost vertices in Γ. Some of these indices are possibly equal. Assume w.l.o.g. that the vertical length of Γ is at least $\epsilon r/\sqrt{2}$. Otherwise, the horizontal length of Γ has this property and we can rotate the descriptions in the argument. The upper halfcircle with centre $X_{i_{\mathsf{T}}}$ and the lower halfcircle with centre $X_{i_{\mathsf{B}}}$ are disjoint and are contained in $\mathcal{S}'$. If $X_{i_{\mathsf{R}}}$ is at greater vertical distance from $X_{i_{\mathsf{T}}}$ than from $X_{i_{\mathsf{B}}}$, consider the rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r - \epsilon r/(2\sqrt{2})$ with one corner on $X_{i_{\mathsf{R}}}$ and above and to the right of $X_{i_{\mathsf{R}}}$. Otherwise, consider the same rectangle below and to the right of $X_{i_{\mathsf{R}}}$. This rectangle is also contained in $\mathcal{S}'$ and its interior does not intersect the previously described halfcircles. Analogously, we can find another rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r-\epsilon r/(2\sqrt{2})$ to the left of $X_{i_{\mathsf{L}}}$ and either above or below $X_{i_{\mathsf{L}}}$ with the same properties. Hence, $\mathsf{Area}(\mathcal{S}') > \left(1 + \frac{\epsilon}{5}\right)\pi r^2$. Let $\mathcal{S}^*$ be the union of all the cells in the tessellation which are fully contained in $\mathcal{S}'$ (see Figure 3, right picture).

Choosing α sufficiently small, we can guarantee that $\mathcal{S}^*$ is topologically connected and has area $\mathsf{Area}(\mathcal{S}^*) \geq (1 + \epsilon/6)\pi r^2$. By removing some extra cells from $\mathcal{S}^*$, we can assume that the number of cells in $\mathcal{S}^*$ is exactly $\lceil \frac{(1+\epsilon/6)\pi}{\alpha^2} \rceil$. Now for each $i,j \in \{1,\dots,n\}$ and each union $\mathcal{S}^*$ of

$\lceil \frac{(1+\epsilon/6)\pi}{\alpha^2} \rceil$ cells that is topologically connected, let $\mathcal{E}_{i,j,\mathcal{S}^*}$ be the following event: $\mathcal{S}^*$ contains no points in $\mathcal{X} \setminus \{X_i, X_j\}$, X_j is at distance at least $2r$ from all the points in $\mathcal{S}^*$; $\widehat{\mathcal{R}}'_j$ contains no points in $\widehat{\mathcal{X}} \setminus \{\widehat{X}_i, \widehat{X}_j\}$; and moreover $\widehat{X}_i \in \widehat{\mathcal{P}}_j$. Notice that if X_j is at distance at least $2r$ from all the points in $\mathcal{S}^*$, then $\pi_1^{-1}(\mathcal{S}^*)$ and $\widehat{\mathcal{R}}'_j$ are disjoint. Hence, $\mathsf{Vol}(\pi_1^{-1}(\mathcal{S}^*) \cup \widehat{\mathcal{R}}'_j) \geq (2+\epsilon/6)\pi r^2$ and

$$\mathbf{P}(\mathcal{E}_{i,j,\mathcal{S}^*}) \leq \left(1 - \mathsf{Vol}(\pi_1^{-1}(\mathcal{S}^*) \cup \widehat{\mathcal{R}}'_j)\right)^{n-2} (2q).$$

Similarly, let $\mathcal{F}_{i,j,\mathcal{S}^*}$ be the following event: $\mathcal{S}^*$ contains no points in $\mathcal{X} \setminus \{X_i, X_j\}$; X_j is at distance at most $2r$ from some point in $\mathcal{S}^*$; and moreover $\widehat{X}_i \in \widehat{\mathcal{P}}_j$. Notice that the probability that X_j is at distance at most $2r$ from some point in $\mathcal{S}^*$ is $O(r^2) = O(\log n/n)$. Hence,

$$\mathbf{P}(\mathcal{F}_{i,j,\mathcal{S}^*}) \leq (1 - \mathsf{Area}(\mathcal{S}^*))^{n-2} O\left(\frac{\log n}{n}\right) (2q).$$

Finally, observe that each of the events $(M_1 > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ and $(M_1 > 0 \wedge B_t > 0)$ implies that either $\mathcal{E}_{i,j,\mathcal{S}^*}$ or $\mathcal{F}_{i,j,\mathcal{S}^*}$ hold, for some $i, j \in \{1, \ldots, n\}$ and some topologically connected union $\mathcal{S}^*$ of cells. Therefore, the probabilities of $(M_{3a} > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ and $(M_{3a} > 0 \wedge B_t > 0)$ are at most

$$\sum_{i,j,\mathcal{S}^*} \mathcal{E}_{i,j,\mathcal{S}^*} + \sum_{i,j,\mathcal{S}^*} \mathcal{F}_{i,j,\mathcal{S}^*} = O\left(\frac{qn}{n^{\epsilon/6}}\right).$$

Part 2. Consider all the possible components in $G(\mathcal{X}; r)$ which are embeddable and have diameter at least $6\sqrt{2}r$. Call them components of *type* 2, and let M_2 denote their number at time t.

We tessellate the torus into square cells of side αr, for some fixed but small enough $\alpha > 0$. Our goal is to show that if $G(\mathcal{X}_t; r)$ has some component of *type* 2, then there exists some topologically connected union $\mathcal{S}^*$ of cells with $\mathsf{Area}(\mathcal{S}^*) \geq (11/5)\pi r^2$ and which does not contain any vertex in $\mathcal{X}$. Then, arguing as in Part 1 before, we conclude that both $\mathbf{P}(M_2 > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ and $\mathbf{P}(M_2 > 0 \wedge B_t > 0)$ are $O\left(qn/(n^{1/5}\log n)\right)$. We now proceed to prove the claim on the union of cells $\mathcal{S}^*$. Given a component Γ of *type* 2 in $G(\mathcal{X}_t; r)$, let $\mathcal{S}'$, i_T and i_B be defined as in Part 1. Repeating the same argument in there but replacing ϵr with $6\sqrt{2}r$, we can assume w.l.o.g. that the vertical distance between X_{i_T} and X_{i_B} is at least $6r$, and claim that the upper halfcircle with centre X_{i_T} and radius r and the lower halfcircle with centre X_{i_B} and radius r must be disjoint and contained in $\mathcal{S}'$. Now, consider the region of points in the torus $[0,1)^2$ with the y-coordinate between that of X_{i_T} and X_{i_B}, and split this region into three horizontal bands of the same width. Observe that each band has width at least $2r$ and hence must contain some vertex of Γ. For each of these bands, pick the rightmost vertex of Γ in the band. We select the right lower quartercircle of radius r centred at the vertex if the vertex is closer to the top of the band, or otherwise the right upper quartercircle. We also perform the symmetric operation and choose three more quartercircles to the left of the leftmost vertices in the three bands. All these six quartercircles together with the two halfcircles previously described are by construction mutually disjoint and contained in $\mathcal{S}'$. Therefore $\mathsf{Area}(\mathcal{S}') \geq (5/2)\pi r^2$. Let $\mathcal{S}^*$ be the union of all the cells in the tessellation which are fully contained in $\mathcal{S}'$. We loose a bit of area compared to $\mathcal{S}'$. However, if α was chosen small enough, we can guarantee that $\mathcal{S}^*$ is topologically connected and also $\mathsf{Area}(\mathcal{S}^*) \geq (11/5)\pi r^2$. This α can be chosen to be the same for all components of *type* 2.

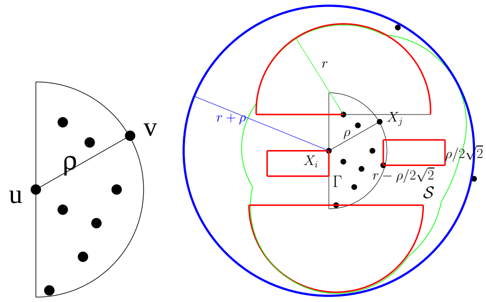

Fig. 4. A component Γ of Type 3 of size exactly $\ell = 9$ and all its vertices at distance $\leq \epsilon r$ from the leftmost one.

Part 3. Consider all the possible components in $G(\mathcal{X}; r)$ which have diameter at most ϵr and size between 2 and $\log n/37$. Call them components of *type* 3, and let M_3 denote their number at time t (see Figure 4, left).

Given any $i \in \{1, \ldots, n\}$, let $\mathcal{E}_i$ be the event that there is a component Γ of *type* 3 in $G(\mathcal{X} \setminus \{X_i\}; r)$ and moreover, for some $j \in \{1, \ldots, n\}$ such that X_j is a vertex of Γ we have that $\widehat{X}_i \in \widehat{\mathcal{P}}_j$. By Theorem 2 of [7], with probability $O(1/\log^2 n)$, $G(\mathcal{X} \setminus \{X_i\}; r)$ has a component Γ of size between 3 and $\log n/37$. Conditional upon this, the probability that $\widehat{X}_i \in \widehat{\mathcal{P}}_j$ for some $j \in \{1, \ldots, n\}$ with $X_j \in \Gamma$ is at most $\log n/37$ times $2q$. This contributes $O(1/\log^2 n)(\log n/37)(2q) = O(q/\log n)$ to the probability of $\mathcal{E}_i$. Otherwise suppose that $G(\mathcal{X} \setminus \{X_i\}; r)$ has a component Γ of *type* 3 and size exactly 2. Again, by Theorem 2 of [7], this happens with probability $O(1/\log n)$. Conditional upon this, the probability that $\widehat{X}_i \in \widehat{\mathcal{P}}_j$ for some $j \in \{1, \ldots, n\}$ with X_j being a vertex of Γ is at most two times $2q$. This also contributes $O(1/\log n)(4q) = O(q/\log n)$ to the probability of $\mathcal{E}_i$, and therefore $\mathbf{P}(\mathcal{E}_i) = O(q/\log n)$.

Given any $i_1, i_2 \in \{1, \ldots, n\}$ ($i_1 \neq i_2$), let $\mathcal{F}_{i_1,i_2}$ be the event that there is a component Γ of *type* 3 in $G(\mathcal{X} \setminus \{X_{i_2}\}; r)$ with $\widehat{\mathcal{R}}'_{i_1} \cap (\widehat{\mathcal{X}} \setminus \{\widehat{X}_{i_1}, \widehat{X}_{i_2}\}) = \emptyset$. To derive the probability of $\mathcal{F}_{i_1,i_2}$, we distinguish two cases according to the distance between X_{i_1} and Γ. Suppose first that for some $h \in \{1, \ldots, n\} \setminus \{i_1, i_2\}$ we have that $r < d(X_{i_1}, X_h) \leq 3r$, which happens with probability $O(r^2) = O(\log n/n)$. Let $\mathcal{S}_h$ be the set of points in $[0,1)^2$ at distance greater than ϵr but at most r from X_h, and let $\mathcal{S}_{i_1}$ be the circle with centre X_{i_1} and radius $r - 2s$. At least one halfcircle of $\mathcal{S}_{i_1}$ has all points at distance greater than r from X_h, so $\mathsf{Area}(\mathcal{S}_h \cup \mathcal{S}_{i_1}) \geq (1-\epsilon^2)\pi r^2 + \pi(r-2s)^2/2 \geq (5/4)\pi r^2$. Notice that, if $\mathcal{F}_{i_1,i_2}$ holds for some component Γ which contains a vertex X_h such that $d(X_{i_1}, X_h) \leq 3r$, then we must have $d(X_{i_1}, X_h) > r$ and moreover $\mathcal{S}_h \cup \mathcal{S}_{i_1}$ must contain no point in $\mathcal{X} \setminus \{X_{i_1}, X_{i_2}\}$, which occurs with probability $(1 - \mathsf{Area}(\mathcal{S}_h \cup \mathcal{S}_{i_1}))^{n-2} = O(1/n^{5/4})$. Multiplying this by the probability that $d(X_{i_1}, X_h) \leq 3r$ and taking the union bound over the $n-2$ possible choices of h, the contribution to $\mathbf{P}(\mathcal{F}_{i_1,i_2})$ due to situations of this type is $O(n(\log n/n)/n^{5/4})$, which is $O(1/(n\log n))$. On the other hand, we claim that the probability that $\mathcal{F}_{i_1,i_2}$ holds for some component Γ with all vertices at distance greater than $3r$ from X_{i_1} is also $O(1/(n\log n))$. To prove this, we first introduce some additional notation: Fix an arbitrary set of indices $J \subset \{1,\ldots,n\}$ of size $|J| = \ell$, with two distinguished elements i and j. Denote by $\mathcal{Y} = \bigcup_{k\in J} X_k$ the set of random points in $\mathcal{X}$ with indices in J, and set $\widehat{\mathcal{Y}} = \pi_1^{-1}(\mathcal{Y})$. Furthermore, let $\mathcal{S}$ be the set of all points in the torus $[0,1)^2$ which are at distance at most r from some vertex in $\mathcal{Y}$, and set $\widehat{\mathcal{S}} = \pi_1^{-1}(\mathcal{S})$. Define $\mathcal{E}$ to be the event that there is some nonnegative real $\rho \le \epsilon r$ such that X_j is placed at distance ρ from X_i and to the right of X_i; all the remaining vertices in $\mathcal{Y}$ are inside the halfcircle of center X_i and radius ρ; and all the $n-\ell-2$ points in $\widehat{\mathcal{X}} \setminus (\widehat{\mathcal{Y}} \cup \{\widehat{X}_{i_1}, \widehat{X}_{i_2}\})$ lie outside of $\widehat{\mathcal{S}} \cup \widehat{\mathcal{R}}'_{i_1}$. This last situation occurs with probability $\widehat{P} = (1-\mathsf{Vol}(\widehat{\mathcal{S}} \cup \widehat{\mathcal{R}}'_{i_1}))^{n-\ell-2}$. By calculations that are analogous to those that yield (4) in the proof of Lemma 4 in [7] (and similar in flavour to Part 1 of this lemma), we obtain

$$\pi r^2\left(2+\frac{1}{6}\frac{\rho}{r}\right) < \mathsf{Vol}(\widehat{\mathcal{S}} \cup \widehat{\mathcal{R}}'_{i_1}) < \frac{13\pi}{4}r^2.$$

Using the fact that $1-x \le e^{-x}$ and plugging in the definition of μ (recall that $\mu = ne^{-r^2\pi n}$), we also get

$$\widehat{P} < \left(\frac{\mu}{n}\right)^{2+\rho/(6r)} \frac{1}{(1-13\pi r^2/4)^{\ell+1}}.$$

Then, one can calculate $\mathbf{P}(\mathcal{E})$ by integrating with respect to ρ the probability density function of $d(X_i,X_j)$ times the probability that the remaining $\ell-2$ selected vertices lie inside the right halfcircle of center X_i and radius ρ times the upper bound on $\widehat{P}$ (again, the calculations are analogous to the last lines of the proof of Lemma 4 of [7], with P from there replaced by $\widehat{P}$), and the claim is proven for components of *type* 3 of fixed size $\ell \ge 2$. By calculating the expected number of components of this type and each size $2 \le k \le \log n/37$ (the argument is as in Part 1 of Lemma 5 of [7], where all details are given) this is extended to all components of *type* 3 and we obtain that $\mathbf{P}(\mathcal{F}_{i_1,i_2}) = O(1/(n\log n))$.

Now we proceed to prove (17) for components of *type* 3. First observe that the event $(M_3 > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ implies that $\mathcal{E}_i$ holds for some $i \in \{1,\ldots,n\}$, since the only way for a component of *type* 3 to disappear within one time step is getting joined to something else. Therefore,

$$\mathbf{P}(M_3 > 0 \wedge \widetilde{K}_{2,t+1} = 0) \le \sum_{i=1}^{n} \mathbf{P}(\mathcal{E}_i) = O\left(\frac{qn}{\log n}\right).$$

Notice that $(M_3 > 0 \wedge B_t > 0)$ implies that $\mathcal{F}_{i_1,i_2}$ holds and moreover $\widehat{X}_{i_2} \in \widehat{\mathcal{Q}}'_{i_1}$, for some $i_1, i_2 \in \{1,\ldots,n\}$ $(i_1 \ne i_2)$. Then,

$$\begin{aligned}\mathbf{P}(M_3 > 0 \wedge B_t > 0) &\le \sum_{i_1,i_2} \mathbf{P}\big(\mathcal{F}_{i_1,i_2} \wedge (\widehat{X}_{i_2} \in \widehat{\mathcal{Q}}'_{i_1})\big) \\ &= O\left(\frac{n^2 q}{n\log n}\right) = O\left(\frac{qn}{\log n}\right).\end{aligned}$$

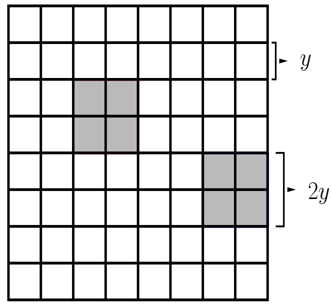

Fig. 5. The tessellation for counting components of *type* 4 with two particular boxes shaded.

Part 4. Consider all the possible components in $G(\mathcal{X};r)$ which have diameter at most ϵr and size greater than $\log n/37$. Call them components of *type* 4, and let M_4 denote their number at time t.

We tessellate the torus with square cells of side $y = \lfloor(\epsilon r)^{-1}\rfloor^{-1}$ ($y \ge \epsilon r$ but also $y \sim \epsilon r$). We define a box to be a square of side $2y$ consisting of the union of 4 cells of the tessellation. Consider the set of all possible boxes. Note that any component of *type* 4 must be fully contained in some box (see Figure 5).

Given any box b and $i,j \in \{1,\ldots,n\}$ $(i \ne j)$, we define $\mathcal{E}_{b,i,j}$ to be the event that box b contains more than $\frac{\log n}{37} - 1$ points of $\mathcal{X} \setminus \{X_i\}$ and moreover $\widehat{X}_i \in \widehat{\mathcal{P}}_j$. Observe that each of the events $(M_4 > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ and $(M_4 > 0 \wedge B_t > 0)$ implies that $\mathcal{E}_{b,i,j}$ holds for some box b and $i,j \in \{1,\ldots,n\}$.

Notice that the number of vertices in each box follows a binomial distribution with mean $\mathbf{E}W = (2y)^2 n = (2\epsilon)^2 \log n/\pi$. Thus, by the Chernoff inequality (see e.g. Theorem 12.7 of [8]), applied with $\delta \sim \frac{\pi}{148\epsilon^2} > e^{79}$ we have

$$\mathbf{P}(W > \frac{1}{37}\log n - 1) < n^{-2.1},$$

and by taking a union bound over the set of all $\Theta(r^{-1}) = \Theta(n/\log n)$ boxes we get $\mathbf{P}(M_4 > 0) = O(1/(n^{1.1}\log n))$ and we get

$$\begin{aligned}\mathbf{P}(M_4 > 0 \wedge \widetilde{K}_{2,t+1} = 0) &\le O\left(\frac{1}{n^{1.1}\log n}\right)\sum_{i,j}\mathbf{P}(\widehat{X}_j \in \widehat{\mathcal{P}}_i) \\ &= O\left(\frac{qn}{n^{0.1}\log n}\right).\end{aligned}$$

The same bound applies to $\mathbf{P}(M_4 > 0 \wedge B_t > 0)$.

Part 5. Consider all the possible components in $G(\mathcal{X};r)$ which are not embeddable and not solitary. Call them components of *type* 5, and let M_5 denote their number at time t. The idea of the proof is the following: We tessellate the torus $[0,1)^2$ into $\Theta(n/\log n)$ small square cells of side length αr, where $\alpha > 0$ is a sufficiently small positive constant (see Figure 6, left). By dividing $[0,1)^2$ into horizontal and vertical bands of width $2r$ and carefully choosing vertices of Γ in each of those bands, one can show that each of the events $(M_5 > 0 \wedge \widetilde{K}_{2,t+1} = 0)$ and $(M_5 > 0 \wedge B_t > 0)$ implies that for some $i,j \in \{1,\ldots,n\}$, there is some connected union $\mathcal{S}^*$ of cells in the tessellation with $\mathsf{Area}(\mathcal{S}^*) \ge (11/5)\pi r^2$ such that $\mathcal{S}^* \cap (\mathcal{X} \setminus \{X_i\}) = \emptyset$, and moreover $\widehat{X}_i \in \widehat{\mathcal{P}}_j$. The proof

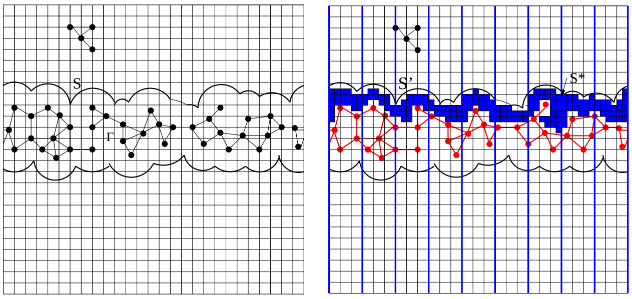

Fig. 6. Components which are not embeddable and not solitary.

is similar to the one in Part 2 of this lemma. From there we obtain $\mathbf{P}(M_5 > 0) = O(\frac{1}{n^{6/5}\log n})$ and therefore we get

$$\mathbf{P}(M_5 > 0 \wedge \widetilde{K}_{2,t+1} = 0) \le O\left(\frac{qn}{n^{1/5}\log n}\right),$$

and the same bound applies to $\mathbf{P}(M_5 > 0 \wedge B_t > 0)$. ■

Now we can characterize the connectivity of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ at two consecutive steps. We denote by $\mathcal{C}_t$ the event that $G(\mathcal{X}_t;r)$ is connected, and by $\mathcal{D}_t = \overline{\mathcal{C}_t}$ the event that $G(\mathcal{X}_t;r)$ is disconnected.

Corollary 8. *Assume that $\mu = \Theta(1)$. Then,*

$$\begin{aligned}
\mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1}) &\sim e^{-\mu}(1 - e^{-\mathbf{E}B}),\\
\mathbf{P}(\mathcal{D}_t \wedge \mathcal{C}_{t+1}) &\sim e^{-\mu}(1 - e^{-\mathbf{E}B}),\\
\mathbf{P}(\mathcal{C}_t \wedge \mathcal{C}_{t+1}) &\sim e^{-\mu}e^{-\mathbf{E}B},\\
\mathbf{P}(\mathcal{D}_t \wedge \mathcal{D}_{t+1}) &\sim 1 - 2e^{-\mu} + e^{-\mu}e^{-\mathbf{E}B}.
\end{aligned}$$

Proof: First observe that $K_{1,t} = S_t + D_t$ and $K_{1,t+1} = S_t + B_t$. Therefore,

$$\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) = \mathbf{P}(S_t = 0 \wedge D_t = 0 \wedge B_t > 0),$$

and by Proposition 6 we get

$$\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) \sim e^{-\mu}(1 - e^{-\mathbf{E}B}). \tag{18}$$

We want to relate this probability with $\mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1})$. In fact, by partitioning $(K_{1,t} = 0 \wedge K_{1,t+1} > 0)$ and $(\mathcal{C}_t \wedge \mathcal{D}_{t+1})$ into disjoint events, we obtain

$$\begin{aligned}
\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) &= \mathbf{P}(\mathcal{C}_t \wedge K_{1,t+1} > 0)\\
&\quad + \mathbf{P}(\mathcal{D}_t \wedge K_{1,t} = 0 \wedge K_{1,t+1} > 0),\\
\mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1}) &= \mathbf{P}(\mathcal{C}_t \wedge K_{1,t+1} > 0)\\
&\quad + \mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1} \wedge K_{1,t+1} = 0),
\end{aligned}$$

and thus we can write

$$\mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1}) = \mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) + P_1 - P_2, \tag{19}$$

where $P_1 = \mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1} \wedge K_{1,t+1} = 0)$ and $P_2 = \mathbf{P}(\mathcal{D}_t \wedge K_{1,t} = 0 \wedge K_{1,t+1} > 0)$.

Suppose that $s = o\big(1/(rn)\big)$. In this case, $\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) = \Theta(srn)$ (see (18) and Proposition 6). Also observe that $\mathcal{D}_t \wedge (K_{1,t} = 0)$ implies that $\widetilde{K}_{2,t} > 0$. In fact, we must have at least two components of size greater than 1, so at least one of these must be non-solitary. Then, we have that $P_1 \le \mathbf{P}(\widetilde{K}_{2,t} = 0 \wedge \widetilde{K}_{2,t+1} > 0)$ and $P_2 \le \mathbf{P}(\widetilde{K}_{2,t} > 0 \wedge B_t > 0)$, and from Lemma 7 we get

$$\begin{aligned}
P_1 &= o\big(\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0)\big) \text{ and}\\
P_2 &= o\big(\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0)\big).
\end{aligned} \tag{20}$$

Otherwise if $s = \Omega\big(1/(rn)\big)$, then $\mathbf{P}(K_{1,t} = 0 \wedge K_{1,t+1} > 0) = \Theta(1)$. In this case, we simply use the fact that $P_1 \le \mathbf{P}(\widetilde{K}_{2,t+1} > 0) = o(1)$ and $P_2 \le \mathbf{P}(\widetilde{K}_{2,t} > 0) = o(1)$, and deduce that (20) also holds.

Finally, the asymptotic expression of $\mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1})$ is obtained from (18), (19) and (20). Moreover, by considering the time-reversed process, we deduce that $\mathbf{P}(\mathcal{D}_t \wedge \mathcal{C}_{t+1}) = \mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1})$. The remaining probabilities in the statement are computed from (1) together with Lemma 2, and using the fact that

$$\begin{aligned}
\mathbf{P}(\mathcal{C}_t \wedge \mathcal{C}_{t+1}) &= \mathbf{P}(\mathcal{C}_t) - \mathbf{P}(\mathcal{C}_t \wedge \mathcal{D}_{t+1}),\\
\mathbf{P}(\mathcal{D}_t \wedge \mathcal{D}_{t+1}) &= \mathbf{P}(\mathcal{D}_t) - \mathbf{P}(\mathcal{D}_t \wedge \mathcal{C}_{t+1}).
\end{aligned}$$

■

For the next lemma, recall the definition of $L_t(\mathcal{C})$ and $L_t(\mathcal{D})$ from Section 2.3. Let $\mathcal{A}$ be an event in the static $G(\mathcal{X};r)$. We denote by $\mathcal{A}_t$ the event that $\mathcal{A}$ holds at time t. In the $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ model, define $L_t(\mathcal{A})$ to be the number of consecutive steps that $\mathcal{A}$ holds starting at step t (possibly 0 if A_t does not hold). The distribution of $L_t(\mathcal{A})$ does not depend on t, and we often omit the t when it is understood.

Lemma 9. *Suppose that $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ (but possibly $\mathbf{E}\left(L(\mathcal{C})\right) \to +\infty$ as $n \to +\infty$). Then conditional upon $\mathcal{C}_t$ but not C_{t-1} we have*

$$\mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right) = \frac{\mathbf{Pr}\left[\mathcal{C}\right]}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]},$$

which does not depend on t. The same statement holds if we interchange $\mathcal{C}$ and $\mathcal{D}$.

Proof: We have that $L_{t-1}(\mathcal{C}) + 1[\mathcal{D}_{t-1}]L_t(\mathcal{C}) = 1[\mathcal{C}_{t-1}] + L_t(\mathcal{C})$, taking expectations and using the hypothesis that $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ we get

$$\mathbf{E}\left(1[\mathcal{D}_{t-1}]L_t(\mathcal{C})\right) = \mathbf{Pr}\left[\mathcal{C}\right], \quad \forall t.$$

The statement follows from the fact that

$$\begin{aligned}
\mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right) &= \frac{\mathbf{E}\left(1[\mathcal{D}_{t-1} \wedge \mathcal{C}_t]L_t(\mathcal{C})\right)}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]}\\
&= \frac{\mathbf{E}\left(1[\mathcal{D}_{t-1}]L_t(\mathcal{C})\right)}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]}.
\end{aligned}$$

■

To prove that $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ and $\mathbf{E}\left(L(\mathcal{D})\right) < +\infty$ we need the following technical lemma.

Lemma 10. *Let $b = b(n)$ be the smallest natural number such that $(b-2)s/3 \ge \sqrt{2}/2$. Then, there exists $p = p(n) > 0$ such that for any fixed circle $\mathcal{R} \subset [0,1)^2$ of radius $r/2$, any $i \in \{1,\ldots,n\}$, any $t \in \mathbb{Z}$, and conditional upon any particular position of $X_{i,t}$ in the torus, the probability that $X_{i,t+b} \in \mathcal{R}$ is at least p.*

Proof: First assume that vertex i changes its angle at each of the b steps following time t. This holds with probability

$(1/m)^b > 0$, and is independent from the initial position and the particular choices of the angles.

Fix an arbitrary position for $X_{i,t} \in [0,1)^2$ and an arbitrary position for circle $\mathcal{R} \subset [0,1)^2$ of radius $r/2$ and center X. Let $Y_k = X_{i,t+k}$ $(0 \le k \le b)$ and denote by α_k the angle in which vertex i moves between Y_k and Y_{k+1}. Recall that each α_k is selected uniformly and independently at random from the interval $[0, 2\pi)$ and that $\mathsf{d}(Y_{k+1}, Y_k) = s$, $\forall k \in \{0, \ldots, b-1\}$.

To prove the statement, we compute a lower bound on the probability of a strategy that is sufficient for vertex i to reach $\mathcal{R}$ at time $t+b$. We start from Y_0 and build a sequence of points $Y_0, \ldots, Y_b$ satisfying the previous conditions and such that $\mathsf{d}(Y_b, X) \le r/2$, by imposing some restrictions on the angles $\alpha_0, \ldots, \alpha_{b-1}$. For the sake of simplicity in the geometrical descriptions, assume that $Y_0, \ldots, Y_b$ and X belong to $\mathbb{R}^2$ and $\mathsf{d}(Y_0, X) \le \sqrt{2}/2$. Once the construction is completed, we map them back to the torus by the usual projection.

For each k, $0 \le k \le b-3$, we restrict α_k to be in $[\theta_k - \pi/6, \theta_k + \pi/6] \pmod{2\pi}$, where θ_k is the angle of $\overrightarrow{Y_k X}$ with respect to the horizontal axis. We claim that, with this choice of angle, $\mathsf{d}(Y_k, X)$ is decreased at each step by at least $s/3$ until it is at most s. By the law of cosines,

$$\mathsf{d}(Y_{k+1}, X) \le \sqrt{\big(\mathsf{d}(Y_k, X)\big)^2 + s^2 - \sqrt{3}\,\mathsf{d}(Y_k, X)\, s}. \tag{21}$$

If $\mathsf{d}(Y_k, X) > s$, we can write

$$\mathsf{d}(Y_{k+1}, X) \le \mathsf{d}(Y_k, X) - \frac{1}{3}s. \tag{22}$$

If $\mathsf{d}(Y_k, X) \le s$, from (21) we deduce that also

$$\mathsf{d}(Y_{k+1}, X) \le \sqrt{s^2 + (1-\sqrt{3})\big(\mathsf{d}(Y_k, X)\big)^2} \le s. \tag{23}$$

From the definition of b, it is easy to see that (21), (22) and (23) imply that $\mathsf{d}(Y_{b-2}, X) \le s$.

Denote by W one of the two points on the perpendicular bisector of $\overline{Y_{b-2}X}$ which satisfy $\mathsf{d}(W, Y_{b-2}) = \mathsf{d}(W, X) = s$. We want to set the angles α_{b-2} and α_{b-1} so that Y_{b-1} and Y_b are close to W, and X, respectively. Indeed, if ϕ_{b-2} and ϕ_{b-1} are the angles between the horizontal axis and, respectively, $\overrightarrow{Y_{b-2}W}$ and $\overrightarrow{WX}$, then by imposing that $\alpha_k \in [\phi_k - \epsilon r/s, \phi_k + \epsilon r/s] \pmod{2\pi}$ for some small enough $\epsilon > 0$, we achieve that $\mathsf{d}(Y_b, X) \le r/2$ and thus $Y_b \in \mathcal{R}$.

Therefore, the probability of choosing all the angles according to the strategy described is $(1/6)^{b-2}\Theta\big((r/s)^2\big)$, and the statement follows with $p = (1/m)^b (1/6)^{b-2}\Theta\big((r/s)^2\big)$. ∎

The next lemma allows us to apply Lemma 9.

Lemma 11. $\mathbf{E}(L(\mathcal{C})) < +\infty$ *and* $\mathbf{E}(L(\mathcal{D})) < +\infty$.

Proof: Fix a circle $\mathcal{R} \subset [0,1)^2$ of radius $r/2$, and take b as in the statement of Lemma 10. Since all vertices choose their angles independently, we have by Lemma 10 that, conditional upon any arbitrary $\mathcal{X}_t$, the probability that all vertices end up inside $\mathcal{R}$ after b steps is $\mathbf{Pr}[\mathcal{X}_{t+b} \subset \mathcal{R} \mid \mathcal{X}_t] \ge p^n$, for some $p = p(n) > 0$. Observe that for any $t \in \mathbb{Z}$ the event $(\mathcal{X}_t \subset \mathcal{R})$ implies that $G(\mathcal{X}_t; r)$ is a clique (and thus connected). Therefore, for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{Pr}\left[\bigwedge_{j=0}^{k} \mathcal{D}_{t+jb}\right] &\le (1-p^n)\mathbf{Pr}\left[\bigwedge_{j=0}^{k-1} \mathcal{D}_{t+jb}\right] \\ &\le (1-p^n)^k \mathbf{Pr}[\mathcal{D}_t].\end{aligned} \tag{24}$$

As $L_t(\mathcal{D}) = \sum_{j=0}^{\infty} 1[\mathcal{D}_t] \cdots 1[\mathcal{D}_{t+j}]$, is satisfied pointwise, for every element in the probability space $(\mathcal{X}_t)_{t \in \mathbb{Z}}$, by the Monotone Convergence Theorem, (24) and the fact that $p > 0$, we conclude

$$\begin{aligned}\mathbf{E}(L_t(\mathcal{D})) &= \sum_{j=0}^{\infty} \mathbf{Pr}[\mathcal{D}_t \wedge \cdots \wedge \mathcal{D}_{t+j}] \\ &\le b \sum_{k=0}^{\infty} \mathbf{Pr}[\mathcal{D}_t \wedge \mathcal{D}_{t+b} \wedge \cdots \wedge \mathcal{D}_{t+kb}] \\ &\le b\,\mathbf{Pr}[\mathcal{D}_t] \sum_{k=0}^{\infty} (1-p^n)^k < +\infty.\end{aligned}$$

A similar argument shows that $\mathbf{E}(L(\mathcal{C})) < +\infty$. ∎

Theorem 1 follows from Lemma 11, Lemma 9 and Corollary 8.

IV. Conclusion.

In this work, we have formally introduced the dynamic random geometric graph in order to study analytically a variation of the Random Walk model for MANETs, defined in [11]. One aim of the present paper was to present a formal framework for highly dynamic networks where the use of ad-hoc data structures is not feasible. We studied the expected length of the connectivity and disconnectivity periods, taking into account different step sizes s and different lengths m during which the angle remains invariant, always considering the static connectivity threshold $r = r_c$. We believe that a similar analysis can be performed for other values of $r \neq r_c$ as well. A different setting to be studied is for the case when the connectivity radii are different for different vertices. It would also be interesting to obtain further information about the connectivity/disconnectivity periods like their variance or their distribution. Another interesting parameter to be studied could be the lengths of the periods it takes (for a given vertex) to reach a certain area of the unit torus (or to remain there, once it has arrived there).

Our model is defined on the unit torus. As mentionned in Section I, an interesting open problem is to compute the connectivity periods on the *unit square* $[0,1]^2$. In this model, each time a vertex touches the boundary of the square, it is forced to change direction (in most models such a vertex is assumed to bounce back). These forced changes seem to make the formal analysis quite more complicated than the one in the present paper. We conjecture that asymptotically the effect of the boundary is negligible, and that the connectivity results for $[0,1]^2$ are asymptotically equivalent to the ones obtained in the present paper.

The *Random Walk* model simulates the behavior of a swarm of mobile vertices as sensors or robots, which move randomly to monitor an unknown territory or to search in it. There exist other models such as the *Random Way-point* model, where each vertex chooses randomly a fixed way-point (from a set of pre-determined way-points) and moves there, and when it arrives it chooses another and moves there (see [5]). A possible line of future research is to do a study similar to the one developed in this paper for this way-point model. We believe that the techniques developed in this paper will prove to be very useful to carry out such a study.

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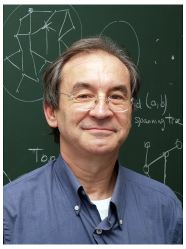
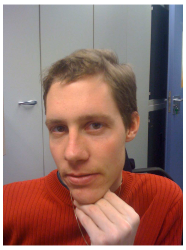


Josep Diaz is Full Professor at the Software Department in the the Technical University of Catalonia (UPC). His areas of interest are in the field of Theoretical Computer Science.



Dieter Mitsche is a visiting Professor at the Technical University of Catalonia (UPC). He obtained his PhD at the Institut für Theoretische Informatik, ETH Zürich. His scientific interests include topics related to randomized techniques in computer science, and discrete mathematics.

Xavier Pérez got his Doctorate in Applied Mathematics at the Technical University of Catalunya (UPC) in 2007. He was awarded a two-year postdoctoral fellowship by the Canadian government, at the University of Waterloo, where he is currently working. His main research interests comprise several fields of discrete mathematics and applied probability (graph theory, asymptotic enumeration, stochastic processes), as well as theoretical computer science (constraint satisfaction problems, algorithms) and statistical physics.