Approximating Layout Problems on Random Graphs *

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Abstract

We show that, with overwhelming probability, several well known layout problems are approximable within a constant for random graphs drawn from the $\mathcal{G}(n, p_n)$ model where $C/n \leq p_n \leq 1$ for all n big enough and for some properly characterized parameter $C > 1$. In fact, our results establish that, with overwhelming probability, the cost of any arbitrary layout of such a random graph is within a constant of the optimal cost.

1 Introduction

Several well-known optimization problems on graphs can be formulated as *graph layout problems.* A *layout* of a graph with n vertices is a bijection between the vertex set and the set of naturals from 1 to n. Graph layout problems, also denoted in the literature as *linear ordering problems* or *linear arrangement problems*, seek for a layout that minimizes a cost associated with each problem. The particular layout problems that we consider are *Bandwidth*, *Minimum Linear Arrangement*, *Cutwidth*, *Modified Cut*, *Sum Cut* (also known as *Profile*) and *Vertex Separation* (equivalent to *Pathwidth*). We also consider the *Edge Bisection* and *Vertex Bisection* problems, which are partitioning problems, but can also be formulated as layout problems. It is well known that these problems have several applications in VLSI, matrix theory, interconnection networks, job scheduling, graph drawing ...

All the considered problems are **NP**-hard. Because of their practical importance, the lack of efficient exact algorithms for general graphs has given rise to the investigation of approximation algorithms or heuristic methods. A standard way of evaluating the *real* efficiency (from a practical point of view) of an algorithm or heuristic is to evaluate its performance on random instances. It is thus natural to start analyzing the approximability properties of layout problems on random graphs, and in particular, on sparse random graphs. For the case of Bandwidth and Edge Bisection, it has already been proved that these problems can be approximated within a constant on random graphs [2, 3, 11]. In this paper, we extend this kind of result to several other layout problems using a unique framework involving "mixing graphs".

Let us first introduce the problems and graphs we consider in this paper:

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Given an undirected graph $G = (V, E)$ with $n = |V|$, a *layout* φ of G is a one-to-one function $\varphi : V \to \{1, \ldots, n\}$. Given a layout φ of G, we define the sets

$$L(i, \varphi, G) = \{u \in V \ : \ \varphi(u) \leq i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V \ : \ \varphi(u) > i\}$$

and the following measures:

$$\begin{aligned} \theta(i, \varphi, G) &= |\{uv \in E \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|, \\ \zeta(i, \varphi, G) &= |\{uv \in E \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \neq i\}|, \\ \delta(i, \varphi, G) &= |\{u \in L(i, \varphi, G) \ : \ \exists v \in R(i, \varphi, G) : uv \in E\}|, \\ \lambda(uv, \varphi, G) &= |\varphi(u) - \varphi(v)| \qquad \text{where } uv \in E. \end{aligned}$$

The problems we address are:

- Bandwidth (BANDWIDTH): Given a graph $G = (V, E)$, find $\text{MINBW}(G) = \min_\varphi \text{BW}(\varphi, G)$ where $\text{BW}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G)$.
- Minimum Linear Arrangement (MINLA): Given a graph $G = (V, E)$, find $\text{MINLA}(G) = \min_\varphi \text{LA}(\varphi, G)$ where $\text{LA}(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G) = \sum_{i=1}^{n} \theta(i, \varphi, G)$.
- Cutwidth (CUTWIDTH): Given a graph $G = (V, E)$, find $\text{MINCW}(G) = \min_\varphi \text{CW}(\varphi, G)$ where $\text{CW}(\varphi, G) = \max_{i=1}^{n} \theta(i, \varphi, G)$.
- Modified Cut (MODCUT): Given a graph $G = (V, E)$, find $\text{MINMC}(G) = \min_\varphi \text{MC}(\varphi, G)$ where $\text{MC}(\varphi, G) = \sum_{i=1}^{n-1} \zeta(i, \varphi, G)$.
- Sum Cut (SUMCUT): Given a graph $G = (V, E)$, find $\text{MINSC}(G) = \min_\varphi \text{SC}(\varphi, G)$ where $\text{SC}(\varphi, G) = \sum_{i=1}^{n} \delta(i, \varphi, G)$.
- Vertex Separation (VERTSEP): Given a graph $G = (V, E)$, find $\text{MINVS}(G) = \min_\varphi \text{VS}(\varphi, G)$ where $\text{VS}(\varphi, G) = \max_{i=1}^{n} \delta(i, \varphi, G)$.
- Edge Bisection (EDGEBIS): Given a graph $G = (V, E)$, find $\text{MINEB}(G) = \min_\varphi \text{EB}(\varphi, G)$ where $\text{EB}(\varphi, G) = \theta(\lfloor n/2 \rfloor, \varphi, G)$.
- Vertex Bisection (VERTBIS): Given a graph $G = (V, E)$, find $\text{MINVB}(G) = \min_\varphi \text{VB}(\varphi, G)$ where $\text{VB}(\varphi, G) = \delta(\lfloor n/2 \rfloor, \varphi, G)$.

Consistently, in the following we will also use the following notation for any layout cost $F \in \{\text{BW}, \text{LA}, \text{CW}, \text{MC}, \text{SC}, \text{VS}, \text{EB}, \text{VB}\}$ and any graph G:

$$\text{MAX}F(G) = \max_\varphi F(\varphi, G) \quad \text{and} \quad \text{AVG}F(G) = \frac{1}{n!} \sum_\varphi F(\varphi, G).$$

Let us recall that, given a positive integer n and a probability p, the class of *random graphs* $\mathcal{G}(n, p)$ is a probability space over the set of undirected graphs $G = (V, E)$ on the vertex set $V = \{1, \ldots, n\}$ determined by $\mathbf{Pr}[uv \in E] = p$ with these events mutually independent.

In the following, we consider random graphs whose edge probability, p_n, is a function of their number of vertices n. In order not having too much connected components, we shall require $C/n \leq p_n \leq 1$ for all $n \geq n_0$ where C is a parameter greater than 1 that will be properly specified latter on and n_0 is some natural. This condition requires the existence of a giant component in the random graph. Recall [1, 2, 5] that random graphs exhibit a phase transition at $p_n = 1/n$: when $p_n = c/n$ with $c < 1$, random graphs consist of many small components, whose largest size is $\Theta(\log n)$; when $p_n = c/n$ with $c > 1$, random graphs turn to have a giant component whose size is $\Theta(n)$.

For any given a constant r, an algorithm A is an *r-approximation* to an optimization problem for a function F when it holds that, for any input x,

$$\frac{1}{1+r} \leq \frac{A(x)}{\textsc{Opt}F(x)} \leq 1+r.$$

Equivalently, the value $\varepsilon = 1 - r$ is called the approximation ratio of the r-approximate algorithm A [7]. Observe that any bound on the gap of a layout cost F gives a bound on the approximation ratio of any algorithm that computes a layout for a graph G. On the other hand, recall that a sequence of events $(\mathcal{E}_n)_{n\geq 1}$ occurs *with high probability* if $\lim_{n\to\infty} \mathbf{Pr}[\mathcal{E}_n] = 1$, and that in the case $\mathbf{Pr}[\mathcal{E}_n] \geq 1 - 2^{-\Omega(n)}$ for all n big enough, we say that $(\mathcal{E}_n)_{n\geq 1}$ occurs *with overwhelming probability.*

Our main result (Theorem 1) states that, with overwhelming probability, all the considered layout problems are approximable within a constant for random graphs drawn from the $\mathcal{G}(n, p_n)$ model where $C/n \leq p_n \leq 1$. In order to prove it, we ask whether there is any relation between the approximability of the maximization versions of our layouts problems, and whether we can infer some consequence for the minimization versions from our understanding of these maximization versions. It thus makes sense to introduce and estimate the gap between the maximum an the minimum costs of a problem for some instance:

Definition 1. For a layout cost $F \in \{\textsc{bw}, \textsc{la}, \textsc{cw}, \textsc{mc}, \textsc{sc}, \textsc{vs}, \textsc{eb}, \textsc{vb}\}$ and any graph G, we define the *gap* of F on G as the ratio between its minimum and maximum values, that is,

$$\textsc{gap}F(G) = \frac{\textsc{max}F(G)}{\textsc{min}F(G)} = 1 + \frac{\textsc{max}F(G) - \textsc{min}F(G)}{\textsc{min}F(G)}.$$

One of our motivations to study random graphs for layout problems was to enable us to analyze heuristics for random sparse graphs. Unfortunately, our results show that *any* algorithm computing a feasible layout, no matter how good or bad, will perform rather well on random graphs, pointing out that such an evaluation may be unworthy for layout problems.

Our results are valid both for sparse and dense graphs, though in the case of random graphs with constant expected degree, we must ensure the existence of a giant component in the random graph [2, 11]. We remark, though, that for dense graphs ($|E| = \Theta(|V|^2)$) polynomial approximation schemes are known for some of the problems [6, 8].

2 Approximation results

We introduce now a class of graphs that captures the properties we need to bound the gaps for our layout problems on random graphs.

Definition 2 (Mixing graphs). Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural n_0. A graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ is said to be *(ϵ, γ, c_n)-mixing* if

$$m \leq (1+\gamma)\tfrac{1}{2}nc_n \tag{1}$$

and for any two disjoint subsets $A, B \subset V$ such that $|A| \geq \epsilon n$ and $|B| \geq \epsilon n$, it is the case that

$$\left|\theta(A,B) - \frac{c_n}{n}|A||B|\right| \leq \gamma\frac{c_n}{n}|A||B|, \tag{2}$$

where $\theta(A, B)$ denotes the number of edges in E having one endpoint in A and another in B.

Observe that this last condition is equivalent to

$$1-\gamma \leq \left. \frac{\theta(A,B)}{|A||B|} \middle/ \frac{c_n}{n} \right. \leq 1+\gamma.$$

Mixing graphs are much related to *expander graphs*. From the many variations of the definition of expander graphs, let us reproduce the one given in [1]: A graph $G=(V,E)$ is an (n,d,c)-expander if it has n vertices, its degree is d, and for every set $W \subset V$ of cardinality $|W| \leq n/2$, the inequality $|\theta(W, V\setminus W)| \geq c|W|$ holds. The main differences in the two definitions are that expanders are required to be bounded-degree graphs and that the expansion condition has to be satisfied for all (not too large) subsets W of vertices, while the mixing condition only involves sets of vertices of $\Omega(n)$ size.

Expander graphs have the property that the largest and second largest eigenvalue of their adjacency matrix are well-separated. This is a sufficient condition for a graph to have good mixing properties.

Lemma 1. Let $G=(V,E)$ be a regular graph of degree d, let λ be the second largest eigenvalue of the adjacency matrix of G. Then for every ϵ, γ such that $\epsilon\gamma > \lambda/d$, G is (ϵ,γ,d)-mixing.

Proof. The Expander Mixing Lemma (see chapter 6 in [1]) states that if G is a d-regular graph and λ is the second largest eigenvalue of the adjacency matrix of G, then for every two subset of vertices A and B,

$$\left|\theta(A,B) - \frac{d}{n}|A||B|\right| \leq \lambda\sqrt{|A||B|}$$

If $|A| \geq \epsilon n$, $|B| \geq \epsilon n$, and $\epsilon\gamma > \lambda/d$ then we have

$$\left|\theta(A,B) - \frac{d}{n}|A||B|\right| \leq \lambda\frac{|A||B|}{\epsilon n} \leq \gamma\frac{d}{n}|A||B|$$

Note also that, since G is d-regular, $|E| = \frac{1}{2}dn$. It follows that, under the hypothesis of the lemma, G is (ϵ,γ,d)-mixing. ☐

The best known explicit construction of expander graphs with strong separation between degree and second largest eigenvalue is due to Lubotzky, Phillips and Sarnak [9]. Their construction yields regular graphs of degree d where the second largest eigenvalue is less than $2\sqrt{d}$. The construction of [9] then gives (ϵ,γ,d)-mixing graphs for every ϵ and γ and with $d = O(\epsilon^{-2}\gamma^{-2})$. See also [4] for additional discussions on mixing graphs, explicit constructions, and applications.

In this paper, our interest in mixing graphs is motivated by the fact that, with overwhelming probability, a random graph from the $\mathcal{G}_{n,C/n}$ model is mixing (for an appropriate choice of the constant C), as proved in Lemma 2 below, and the fact that the layout problems defined in the previous section are easy to approximate on mixing graphs, as proved in Lemma 3 below. Together, the two results imply that layout problems are easy to approximate on random graphs.

Lemma 2. Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural n_0. Then, for all $n \geq n_0$, random graphs drawn from $\mathcal{G}(n,p_n)$ with $p_n = c_n/n$ are (ϵ,γ,c_n)-mixing with probability at least $1-2^{-\Omega(n)}$.

Proof. Let $n \geq n_0$ and consider a random graph $G = (V, E)$ drawn from $\mathcal{G}(n, p_n)$. Let M be a random variable counting its number of edges. We estimate first the probability that condition (1) fails. The expected number of edges of G is $\mathbf{E}[M] = \frac{1}{2}(n-1)c_n$. Using Chernoff's bounds [10] we get

$$\mathbf{Pr}\left[M > (1+\gamma)\frac{1}{2}nc_n\right] \leq \mathbf{Pr}[M > (1+\gamma)\mathbf{E}[M]] \leq \exp\left(-\tfrac{1}{3}\gamma^2\mathbf{E}[M]\right)$$

$$\leq \exp\left(-\tfrac{1}{6}\gamma^2(n-1)c_n\right) \leq 2^{-\Omega(n)}$$

by the hypothesis $c_n \geq C_{\epsilon,\gamma} > 1$.

We estimate now the probability that condition (2) does not hold. Consider any two disjoint sets $A, B \subset V$ such that $|A|, |B| \geq \epsilon n$. There are $k = |A||B|$ possible edges having an endpoint in A and an endpoint in B. Let us call $X_1, \ldots, X_k$ the Bernoulli random variables such that $X_i = 1$ if the i-th (in lexicographical order) of such edges is in the graph, and $X_i = 0$ otherwise. The expectation of $\sum_{i=1}^k X_i$ is $\mu_n = |A||B|c_n/n$. Using Chernoff's bounds we have that

$$\mathbf{Pr}\left[(1-\gamma)\mu_n < \sum_{i=1}^{k} X_i < (1+\gamma)\mu_n\right] \geq 1 - 2\exp\left(-\tfrac{1}{3}\gamma^2\mu_n\right).$$

Any vertex can either be in A, in B, or not in A and not in B. Therefore, there are at most 3^n choices for the sets A and B. By Boole's inequality it follows that the probability that condition (2) is not satisfied is at most

$$3^n \cdot 2\exp\left(-\tfrac{1}{3}\gamma^2\mu_n\right) \leq \exp\left(\ln 2 + n\ln 3 - \tfrac{1}{3}\gamma^2\epsilon^2 nc_n\right) \leq 2^{-\Omega(n)}$$

by the hypothesis $c_n \geq C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$.

Overall, by Boole's inequality, the failure probability of the conditions can be estimated by the sum of the individual failure probabilities, which therefore is $2^{-\Omega(n)}$. $\square$

Lemma 3. Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural n_0. Let $n \geq n_0$ and let $G = (V, E)$ be any (ϵ, γ, c_n)-mixing graph with $|V| = n$. Then,

$$\textsc{gapcw}(G) \leq 2(\gamma+1)(1-\gamma)^{-1} + O\left(n^{-1}\right), \tag{g1}$$
$$\textsc{gapeb}(G) \leq 2(\gamma+1)(1-\gamma)^{-1} + O\left(n^{-1}\right), \tag{g2}$$
$$\textsc{gapla}(G) \leq (\gamma+1)(1-6\epsilon)^{-1}(1-\gamma)^{-1}, \tag{g3}$$
$$\textsc{gapmc}(G) \leq (\gamma+1)(1-12\epsilon)^{-1}(1-\gamma)^{-1}, \tag{g4}$$
$$\textsc{gapvs}(G) \leq (\tfrac{1}{2}-\epsilon)^{-1} + O\left(n^{-1}\right), \tag{g5}$$
$$\textsc{gapvb}(G) \leq (\tfrac{1}{2}-\epsilon)^{-1} + O\left(n^{-1}\right), \tag{g6}$$
$$\textsc{gapsc}(G) \leq (1-4\epsilon)^{-1}, \tag{g7}$$
$$\textsc{gapbw}(G) \leq (1-2\epsilon)^{-1} + O\left(n^{-1}\right). \tag{g8}$$

Proof. Let $m = |E|$. Let us start proving (g1). Consider any layout φ of G. Take $A = \{u \in V \mid \varphi(u) \leq \lfloor n/2 \rfloor\}$ and $B = V \setminus A$. Then $|A|, |B| \geq n/2 - 1 \geq \epsilon n$. So,

$$\textsc{cw}(\varphi, G) \geq \theta\left(\lfloor n/2 \rfloor \varphi, G\right) \geq \theta(A, B) \geq (1-\gamma)\frac{c_n}{n}|A||B| \geq (1-\gamma)\frac{c_n}{n}\left(n/2-1\right)^2.$$

Therefore, $\text{MINCW}(G) \geq (1-\gamma)c_n(n/2-1)^2/n$. The number of edges of G is an upper bound on $\text{MAXCW}(G)$. As G is a (ϵ, γ, c_n)-mixing graph, $\text{MAXCW}(G) \leq |E| \leq (1+\gamma)\frac{1}{2}(n-1)c_n$. The computation of the gap yields $\text{GAPCW}(\varphi, G) \leq 2\frac{\gamma+1}{1-\gamma} + O\left(n^{-1}\right)$. The proof of (g2) is the same.

Let us prove (g3). Let φ be any layout of G. We have

$$\text{LA}(\varphi, G) \geq \sum_{i=1}^{n} \theta(i, \varphi, G) \geq \sum_{i=\epsilon n}^{(1-\epsilon)n} \theta(i, \varphi, G) \geq (1-\gamma)\frac{c_n}{n} \sum_{i=\epsilon n}^{(1-\epsilon)n} i(n-i).$$

On the other hand, the sum of the cuts at the first ϵn and the last $(1-\epsilon)n$ positions is, each one, not larger than ϵnm. So,

$$\text{LA}(\varphi, G) \leq \sum_{i=1}^{n} \theta(i, \varphi, G) \leq 2\epsilon nm + \sum_{i=\epsilon n}^{(1-\epsilon)n} \theta(i, \varphi, G) \leq 2\epsilon nm + (1+\gamma)\frac{c_n}{n} \sum_{i=\epsilon n}^{(1-\epsilon)n} i(n-i).$$

Therefore, letting $S = c_n n^{-1} \sum_{i=\epsilon n}^{(1-\epsilon)n} i(n-i)$, we have

$$\text{MINLA}(G) \geq (1-\gamma)S,$$
$$\text{MAXLA}(G) \leq 2\epsilon nm + (1+\gamma)S.$$

The average length of an edge on a graph with n vertices is $(n+1)/3$. Taking into account that $\text{LA}(\varphi, G)$ is the sum of all edge lengths, we have $\text{AVGLA}(G) \geq m(n+1)/3$, and therefore, there is a layout that gives at least this value. So, $2\epsilon nm + (1+\gamma)S \geq m(n+1)/3 > mn/3$ and thus $2\epsilon nm \leq \frac{6\epsilon}{(1-6\epsilon)}(1+\gamma)S$. As a consequence,

$$\text{GAPLA}(G) \leq 1 + \frac{\frac{6\epsilon}{(1-6\epsilon)}(1+\gamma)S + 2\gamma S}{(1-\gamma)S} \leq \frac{\gamma+1}{(6\epsilon-1)(\gamma-1)}.$$

Let us prove (g4) in a similar way. Let φ be any layout of G. We have

$$\text{MC}(\varphi, G) \geq \sum_{i=1}^{n-1} \zeta(i, \varphi, G) \geq \sum_{i=\epsilon n+1}^{(1-\epsilon)n-1} \zeta(i, \varphi, G) \geq (1-\gamma)\frac{c_n}{n} \sum_{i=\epsilon n+1}^{(1-\epsilon)n-1} (i-1)(n-i).$$

For the lower bound, we have

$$\begin{aligned}\text{MC}(\varphi, G) &\leq \sum_{i=1}^{n-1} \zeta(i, \varphi, G) \leq 2(\epsilon n+1)m + \sum_{i=\epsilon n+1}^{(1-\epsilon)n-1} \zeta(i, \varphi, G) \\ &\leq 2\epsilon nm + (1+\gamma)\frac{c_n}{n} \sum_{i=\epsilon n+1}^{(1-\epsilon)n-1} (i-1)(n-i).\end{aligned}$$

where the last inequality holds because $\zeta(1, \varphi, G) = \zeta(n, \varphi, G) = 0$ for any layout φ. Therefore, letting $T = c_n n^{-1} \sum_{i=\epsilon n+1}^{(1-\epsilon)n-1} (i-1)(n-i)$, we have

$$\text{MINMC}(G) \geq (1-\gamma)T,$$
$$\text{MAXMC}(G) \leq 2\epsilon nm + (1+\gamma)T.$$

As $\textsc{avgmc}(G) = m(n-5)/3$, there is a layout that gives at least this value. So, $2\epsilon nm + (1+\gamma)T \geq m(n-5)/3 > mn/6$ and thus $2\epsilon nm \leq \frac{12\epsilon}{(1-12\epsilon)}(1+\gamma)T$. As a consequence,

$$\textsc{gapmc}(G) \leq 1 + \frac{\frac{12\epsilon}{(1-12\epsilon)}(1+\gamma)T + (1+\gamma)T)}{(1-\gamma)T} \leq \frac{\gamma+1}{(12\epsilon-1)(\gamma-1)}.$$

Let us prove (g5). Consider any layout φ of G. Notice that in a (ϵ, γ, c_n)-mixing graph it is the case that $\delta(i, \varphi, G) \geq i - \epsilon n$ for every $\epsilon n < i < (1-\epsilon)n$. This is because there cannot be ϵn vertices on the left of i and ϵn vertices on the right of i without any connection. Thus,

$$\textsc{vs}(\varphi, G) \geq \delta(\lfloor n/2 \rfloor, \varphi, G) \geq \lfloor n/2 \rfloor - \epsilon n \geq \left(\tfrac{1}{2} - \epsilon\right) n - 1$$

and therefore $\textsc{minvs}(G) \geq \left(\frac{1}{2} - \epsilon\right) n - 1$. Obviously $\textsc{maxvs}(G) \leq n$, so $\textsc{gapvs} \leq \left(\frac{1}{2} - \epsilon\right)^{-1} + O\left(n^{-1}\right)$. The proof of (g6) is the same.

Let us prove (g7). Again, consider any layout φ of G. We have,

$$\begin{aligned}\textsc{sc}(\varphi, G) &\geq \sum_{i=1}^{n-1} \delta(i, \varphi, G) \geq \sum_{i=\epsilon n}^{(1-\epsilon)n} \delta(i, \varphi, G) \geq \sum_{i=\epsilon n}^{(1-\epsilon)n} (i - \epsilon n) \\ &\geq n^2/2 - 2\epsilon n^2 + n(1-\epsilon)/2 + 3\epsilon^2 n^2 \geq n^2/2 - 2\epsilon n^2.\end{aligned}$$

Therefore, $\textsc{minsc}(G) \geq n^2/2 - 2\epsilon n^2$. Moreover,

$$\textsc{sc}(\varphi, G) \leq \sum_{i=1}^{n-1} \delta(i, \varphi, G) \leq \sum_{i=1}^{n-1} i \leq (n-1)n/2 \leq n^2/2.$$

Therefore, $\textsc{maxsc}(G) \leq n^2/2$. Thus, $\textsc{gapsc} \leq (1-4\epsilon)^{-1}$.

Finally, let us prove (g8). It is clear that $\textsc{maxbw}(G) < n$. Consider any layout φ of G. As G is a (ϵ, γ, c_n)-mixing graph, there must be some edge between the $\lceil \epsilon n \rceil$ first vertices of φ and the $\lfloor (1-\epsilon)n \rfloor$ last vertices of φ. The length of this edge must be at least $(1-2\epsilon)n - 4$, and so $\textsc{minbw}(G) \geq (1-2\epsilon)n - 4$. Thus $\textsc{gapbw}(G) \leq (1-2\epsilon)^{-1} + O\left(n^{-1}\right)$. □

The combination of Lemmas 2 and 3 implies our main result:

Theorem 1. Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1 + \ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n \geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural n_0. Then, with overwhelming probability, the problems Bandwidth, MinLA, Cutwidth, ModCut, SumCut, VertSep, EdgeBis and VertBis, can be approximated within a constant on random graphs $\mathcal{G}(n, p_n)$.

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