Random Geometric Graphs.

Josep Díaz
Random Geometric Graphs

- *Random Euclidean Graphs*,
- *Random Proximity Graphs*,
- Random Geometric Graphs.
Random Euclidean Graphs

Choose a sequence \( V = \{x_i\}_{i=1}^n \) of independent and uniformly distributed (iid) points on \([0, 1]^d\), and consider the weighted complete graph on \( V \), where the weight of an edge is its Euclidean distance.
Random Euclidean Graphs

Choose a sequence $V = \{x_i\}_{i=1}^n$ of independent and uniformly distributed (iid) points on $[0, 1]^d$, and consider the weighted complete graph on $V$, where the weight of an edge is its Euclidean distance.

Random Proximity Graphs.

Choose a sequence $V = \{x_i\}_{i=1}^n$ of independent and uniformly distributed points on $[0, 1]^2$ and a given $k$ (could be a Poisson point process with intensity 1). Consider the graph $G_{n,k}$ with vertex set $V$, and each $v \in V$ connects to its $k$ nearest neighbors. These graphs are also denoted $k$-NNG.
Random Proximity Graphs.

Choose a sequence \( V = \{ x_i \}_{i=1}^n \) of i.i.d. points on \([0, 1]^2\) and a given \( k \). Consider the graph \( G_{n,k} \) with vertex set \( V \), and each \( v \in V \) connects to its \( k \) nearest neighbors.
Connectivity of Undirected Random Proximity graphs

**Xue, Kumar, (2004)**

*\( G_{n,k} \) is connected aas if \( k \geq 0.074 \log n \)

*\( G_{n,k} \) is disconnected aas if \( k \leq 5.1774 \log n \)

**Conjecture:** threshold for connectivity \( k = (1 + o(1)) \log n \).
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$G_{n,k}$ is connected aas if $k \geq 0.3043 \log n$
$G_{n,k}$ is disconnected aas if $k \leq 0.5139 \log n$
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**Balister, Bollobas, Sarkar, Walters, (2006)**

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**Connectivity for directed Random Proximity.**  

**Balister, Bollobas, Sarkar, Walters, (2006)**

$G_{n,k}$ is disconnected aas if $k \geq 0.7209 \log n$  
$G_{n,k}$ is connected aas if $k \leq 0.9967 \log n$
Birth of giant component in undirected $k$-NNG.

*Shang-Hua Teng, F. Yao (2004)*
Empirically, the birth of a giant component occurs at $k = 3$
Analytically the 213-NNG has a giant component.

Open problem: The giant component appears at 3-NNG
Random geometric graphs (RGG)

\(G(n, r)\): Choose a sequence \(\mathcal{V}_n = \{x_i\}_{i=1}^n\) of independent and uniformly distributed points on \([0, 1]^d\), given a fixed \(r(n) > 0\), connect two points if their \(\ell_p\)-distance is at most \(r\).
Random geometric graphs (RGG): Uniformly distributed

$G(n, r)$: Choose a sequence $\mathcal{V}_n = \{x_i\}_{i=1}^n$ of independent and uniformly distributed points on $[0, 1]^d$, given a fixed $r(n) > 0$, connect two points if their $\ell_p$-distance is at most $r$. 
Random geometric graphs (RGG): Uniformly distributed

$G(n, r)$: Choose a sequence $\mathcal{V}_n = \{x_i\}_{i=1}^n$ of independent and uniformly distributed points on $[0, 1]^d$, given a fixed $r(n) > 0$, connect two points if their $\ell_p$-distance is at most $r$. 
Same result starting from a Poisson point process in the square $\sqrt{n} \times \sqrt{n}$.

*E.N. Gilbert: Random Plane Networks.*

*M.D. Penrose: Random Geometric Graphs OUP (2003).*
The deterministic version: Unit Disk Graphs.

*Intersection model:* An undirected graph is a unit disk graph (UDG) if its vertices can be put in one-to-one correspondence with circles of equal radius in the plane in such a way that two vertices are joined by an edge iff the corresponding circles intersect (wlog we can assume the radius of each disk is 1).

*Golumbic:* Algorithmic graph theory and perfect graphs (1980).

There are other equivalent models as the *containment model.*

*Clark, Colbourn, Johnson (1990).*
The recognition problem for UDG.

Given an undirected graph \( G \), decide whether \( G \) is UDG.

*Breu, Kirkpatrick (1998).* The recognition problem for UDG is NP-hard.
An important difference between RGG and $G_{n,p}$. 

![Diagram showing two graphs, one with a dashed line to illustrate the difference between RGG and $G_{n,p}$](image-url)
An important difference between RGG and $G_{n,p}$.

Given a RGG, not all graphs are feasible to be subgraphs of the RGG:

[Diagram showing two graphs, one feasible and one not feasible as subgraphs.]
Basic facts about RGG

In $[0, 1]^2$ and the $\ell_2$-norm.

Recall: if $\forall k \geq 1$, a rv $X$ has $E[X]^k = \mu^k$ for $\mu = \Theta(1)$ then asymptotically $X$ is a Poisson with mean $\mu$.

Let $X = \text{the number of isolated vertices}$ in $G(n, r(n))$.

Consider the indicator r.v. $X_i = 1$ iff vertex $x_i$ is isolated. Then

$$E[X] = \sum_{i=1}^n \Pr[X_i = 1] = n(1 - \pi r(n)^2)^{n-1} \sim ne^{-\pi r(n)^2}n - O(r(n)^4n)$$

Let $\mu = ne^{-\pi r^2 n}$, then

- if $\mu = o(1) \iff E[X] = o(1)$
- if $\mu = \Omega(1) \Rightarrow E[X] \sim \mu$. 
Connectivity of RGG

On $[0, 1]^2$ and $\ell_2$, the asymptotic behavior of $\mu = ne^{-\pi r(n)^2 n}$ characterizes the connectivity of $G(n, r(n))$:

1. If $\mu \to 0$ then $G(n, r(n))$ is aas connected.
2. If $\mu = \Theta(1)$ then $G(n, r(n))$ has a component of size $\Theta(n)$ and $X$ follows a Poisson distribution with parameter $\mu$.
3. If $\mu \to \infty$ then $G(n, r(n))$ is aas disconnected.

Connectivity: when $\mathbb{E}[X] = \Theta(1) \Rightarrow r_c = \sqrt{\frac{\ln n - \ln \mu}{\pi n}}$. $r_c$ is the connectivity radius.
\( \mathcal{C} = \) event \( G(n, r(n)) \) is connected
\( \mathcal{D} = \) event \( G(n, r(n)) \) is disconnected

\[
\Pr[\mathcal{C}] \sim \Pr[X = 0] \sim e^{-\mu} = \Theta(1)
\]
\[
\Pr[\mathcal{D}] \sim \Pr[X > 0] \sim 1 - e^{-\mu}.
\]

M.D. Penrose (1997)
P. Gupta, P.R. Kumar (1998)
A graph property is *monotone* if it is preserved when edges are added to the graph. Ex: connectivity, Hamiltonicity.

*Goel, Rai, Krinashmachari (2004)* Any monotone graph property of a random geometric graph has a sharp threshold at

\[ r = \Theta\left(\frac{\log n}{\alpha n}\right)^{1/d}. \]

where \( \delta = o(1) \) (\( \ln d = 2 \), \( \delta = r_c \ln^{1/4} n \))
$r_c \sim \left( \sqrt{\frac{\ln n}{\pi n}} \right)$ is a sharp threshold for the connectivity of $G(n, r(n))$ on $[0, 1)^2$.

In general for any $\ell_p$, $1 \leq p \leq \infty$, with any $d \geq 2$

$$\left( \frac{\ln n}{\alpha_{p,d} n} \right)^{1/d},$$

for constant $\alpha_{p,d}$. 
Basic facts of RGG.

The *thermodynamical limit* $r_t = \sqrt{\frac{\lambda_c}{n}}$, where whp there appears a giant component in $G(n, r(n))$.

Experimentally for dim=2, $\lambda_c = 2.0736$

For $r_i \leq r_t$, then Prob. $\nu$ is isolated $= 1 - o(1)$. *Most vertices are isolated.*
Feasible subgraphs of a RGG.

Given a feasible graph $\tilde{G}$ of size $k$, how many copies of $\tilde{G}$ will appear as an induced subgraph in $G(n, r(n))$, as $r(n)$ evolves? How many of those copies form a component in $G(n, r(n))$?
Feasible subgraphs of a RGG.

Given a feasible graph $\tilde{G}$ of size $k$, how many copies of $\tilde{G}$ will appear as an induced subgraph in $G(n, r(n))$, as $r(n)$ evolves? How many of those copies form a component in $G(n, r(n))$?

What will be the probability of existence of components of size $i \geq 2$ at connectivity threshold $r_c$? How do they look like?
Induced components.

Ch. 3 of M. Penrose RGG, 2002

Given a feasible $G'$ of size $k$: Let $J = \#$ induced copies of $G'$ in $G(n, r(n))$, which are a component.

For $r(n) < r_t$, then $\mathbb{E}[J] = \Theta(r(n)^{2(k-1)} n^k)$,

Therefore, as $G'$ appear at $G(n, r(n))$ as induced component.
Components above $r_t$

Recall if $\mu = \Theta(1) \Rightarrow$ the probability $G(n, r(n))$ has some component of size $> 1$, other than a giant component, is $o(1)$. 

*D., Mitsche, Pérez (2007)* gave more accurate bounds for that probability:

Given a component of size $i$, at $r_c$ the probability of existence of the component is $\Theta(\frac{1}{\log^{i-1} n})$, aas.

Moreover, aas the components of size $i$ will be cliques.
Let $D$ be a rv counting the degree in $G(n, r(n))$. Then, the \textit{average degree} $\mathbb{E}[D] = \pi r(n)^2(n - 1)$ which at $r_c$ is $\Theta(\ln n)$. As $n \to \infty$, the value of $r(n)$ in the ratio $\frac{nr(n)^2}{\ln n}$ gives us an indication whether $\mathbb{E}[D]$ grows slower or faster than $\ln n$. 
Subconnectivity and superconnectivity regimes

\( G(n, r(n)) \) is:

**Sparse** if \( nr(n)^2 = \mathcal{o}(\ln n) \) and \( nr(n)^2 = n^{o(1)} \) as \( n \to \infty \).

\[
\quad n^{-c} \ll nr(n)^2 \ll \ln n \quad \forall c > 0
\]

Then asymptotically \( G(n, r(n)) \) is disconnected and \( E[D] \) grows slower than \( \ln n \)
Subconnectivity and superconnectivity regimes

\[ G(n, r(n)) \text{ is:} \]

\textbf{Sparse} if \( nr(n)^2 = o(\ln n) \) and \( nr(n)^2 = n^{o(1)} \) as \( n \to \infty \).

\[ n^{-c} << nr(n)^2 << \ln n \quad \forall c > 0 \]

Then asymptotically \( G(n, r(n)) \) is disconnected and \( E[D] \) grows slower than \( \ln n \)

\textbf{Dense} if \( nr(n)^2 / \ln n \to \infty \) as \( n \to \infty \).

\[ nr(n)^2 >> \ln n \]

Then asymptotically \( G(n, r(n)) \) is connected and \( E[D] \) grows faster than \( \ln n \).
Dense and Sparse RGG on $[0, 1]^2$. 
Chromatic number and clique number

Given $G(n, r(n))$ the chromatic number $\chi(G(n, r(n)))$ is the minimum number of colors needed to color the vertices of $G(n, r(n))$, such that no edge shares the same color. The clique number $\omega(G(n, r(n)))$ is the size of the largest clique in $G(n, r(n))$.

*Clark, Colbourn, Johnson, (1990).*

The problem of finding the chromatic number is NPC on UDG
The problem of finding max cliques is poly-time on UDG
Chromatic number and clique number

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The problem of finding the chromatic number is NPC on UDG
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The chromatic and clique numbers on RGG:
$\chi(G(n, r_c)) \sim \Theta(\log n)$ and $\omega(G(n, r_c)) \sim \Theta(\log n)$. 
Chromatic and clique numbers

McDiarmid, Muller (2008)
(Version for \( \ell_2 \) and \([0, 1]^2\))

Given \( G(n, r(n)) \), there is a constant \( 0 < t_0 < \infty \) and a continuous increasing function \( x : [0, \infty] \rightarrow [0, \infty) \), with \( x(t) = 1.103 \) as \( t \rightarrow \infty \) s.t.

if \( nr^2 / \ln n \rightarrow t \) as \( n \rightarrow \infty \), then

\[
\frac{\chi(G(n, r(n)))}{\omega(G(n, r(n)))} \rightarrow x(t) \text{ aas}
\]
Hamiltonicity

Given $G(n, r(n))$ a Hamiltonian cycle is a simple cycle that visits every node of $G(n, r(n))$ exactly once. $G(n, r(n))$ is Hamiltonian if it contains a Hamiltonian cycle.
Hamiltonicity

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*Itai, Papadimitriou, Szwarafiter, (1982)*: The problem of deciding if a UDG is Hamiltonian is NPC.
Hamiltonicity

*M. Penrose RGG (2003):*
Open problem: At the exact point where $G(n, r(n))$ becomes 2-connected, it also becomes Hamiltonian, aas
Hamiltonicity

*M. Penrose RGG (2003):*
Open problem: At the exact point where $G(n, r(n))$ becomes 2-connected, it also becomes Hamiltonian, aas

*D., Mitsche, Pérez (2007):*
The property that $G(n, r(n))$ is Hamiltonian exhibits a sharp threshold at $r(n) = \sqrt{\frac{\ln n}{\pi n}} (1 + o(1))$.

In fact: for any $\epsilon > 0$,

- if $r = \sqrt{\frac{\log n}{(\pi+\epsilon)n}}$, then a.a.s. $G$ contains no Hamiltonian cycle,
- if $r = \sqrt{\frac{\log n}{(\pi-\epsilon)n}}$, then a.a.s. $G$ contains a Hamiltonian cycle.
Bisection

Given $G(n, r(n))$ we want to partition the nodes into two equal cardinality subsets, such that:

* **Min-Bisection:** it minimizes the number of edges crossing between subsets.
* **Max-Bisection:** it maximizes the number of edges crossing between subsets.
Max and Min Bisection

MinB = 4

MaxB = 14
Max-Bisection:

*D., Kaminski. (2007)*: Max-Bisection for UDG is NPC
(Also Max-Cut for UDG is NPC)
Bisection

Max-Bisection:
*D., Kaminski. (2007)*: Max-Bisection for UDG is NPC (Also Max-Cut for UDG is NPC)

Min-Bisection:
*Arora, Karger, Karpinski STOC (1995)*: APX for UDG

Open problem: Is Min-Bisection NPC for UDG?

*D., Penrose, Petit, Serna (1999)*:
If Min-Bisection is NPC when restricted to planar graphs with $\Delta = 4$, then Min-Bisection is NPC for UDG.
Min-Bisection on dense RGG on $[0, 1]^2$

*D., Penrose, Petit, Serna (1999):*

The projection algorithm
For dense $G(n, r(n))$, the size of the Min-Bisection is $\geq cn^2 r(n)^2$. As the ratio of approximation for the projection algorithm is

$$\frac{\text{Min-Bis}(G(n, r(n)))}{n^2 r(n)^3} \sim 0.265$$

As the projection algorithm is a constant approximation for Min-Bisection on dense RGG.
A generalization of Min-Bisection. Given a $G(n, r(n))$ on $[0, 1]^2$ and a $\beta \in (0, 1/2]$, a $\beta$-balanced Min-Cut is a partition of the vertices into 2 subsets, each containing at least $\beta n$ vertices, such that it minimizes the number of edges across the partition. When $\beta = 1/2$ then Min-Bisection
A generalization of Min-Bisection. Given a $G(n, r(n))$ on $[0, 1]^2$ and a $\beta \in (0, 1/2]$, a $\beta$-balanced Min-Cut is a partition of the vertices into 2 subsets, each containing at least $\beta n$ vertices, such that it minimizes the number of edges across the partition. When $\beta = 1/2$ then Min-Bisection

*D., Grandoni, Marchetti (2006)*

Given a $G(n, r(n))$, for at he connectivity threshold, the $\beta$-balanced Min-Cut of $G(n, r(n))$ contains

$\Omega(\min\{\beta n c \log n, \sqrt{\beta n c^3 \log^3 n}\})$ edges, aas.
Models for simulating mobility in Ad Hoc networks:

- Random walk model (R.Guerin-87)
- Random waypoint model (Broch, Maltz, Johnson, Hu, Jetcheva-98)
- The Realistic model (Jardosh, Belding, Almeroth, Suri (2003))
The random walk model for RGG

D., Mitsche, Pérez (2007) (SODA-2008)

In $[0, 1)^2$, let $s = s(n) \in \mathbb{R}^+$ and $M = m(n) \in \mathbb{N}$, at $t = 0$ distribute $n$ vertices on $[0, 1)^2$ iid.
Take $r(n) = r_c$ and construct an initial random geometric graph $G(n, r_c)$.
Each $v$ chooses independently a random angle $\alpha \in [0, 2\pi)$, advancing in that direction a length $s$ (step).
After each step, each vertex independently flips a coin, and with prob $1/m$ they choose a new direction $\alpha$, otherwise it continues in the same direction.
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At $t = k$, let $G_k(n, r_c, s, m)$ be the resulting random geometric graph.
D., Mitsche, Pérez (2007) (SODA-2008)

In \([0, 1)^2\), let \(s = s(n) \in \mathbb{R}^+\) and \(M = m(n) \in \mathbb{N}\), at \(t = 0\) distribute \(n\) vertices on \([0, 1)^2\) iid.

Take \(r(n) = r_c\) and construct an initial random geometric graph \(G(n, r_c)\).

Each \(v\) chooses independently a random angle \(\alpha \in [0, 2\pi)\), advancing in that direction a length \(s\) (step).

After each step, each vertex independently flips a coin, and with prob \(1/m\) they choose a new direction \(\alpha\), otherwise it continues in the same direction.

At \(t = k\), let \(G_k(n, r_c, s, m)\) be the resulting random geometric graph.

*How long the connectivity and disconnectivity periods will last?*
Example: Initial distribution and RGG
Example: Choose initial direction

m=3

r

s

m=3
Example: Move one step $s$
Example: Move one step $s$
Example: step 2 step s

With prob 1/3 change direction
Example: step 2 step s

$t=2$

$r \quad s \quad m=3$
Example: step 2 step s
Example: step 3 step s

\[ t=2 \]

\[ r \quad s \quad m=3 \]
Example: step 3 step s
Example: step 3 step s
Example: step 3 step s
Main result

\[ C_k = \text{event that } G_k(n, r, s, m) \text{ is connected at step } k, \]
\[ D_k = C_k = \text{the event that } G_k(n, r, s, m) \text{ is disconnected at step } k, \]
\[ L_k(C) = \text{the number of consecutive steps that } G(n, r(n)) \text{ remains connected starting at step } k, \]
\[ L_k(D) = \text{the number of consecutive steps that } G(n, r(n)) \text{ remains disconnected starting at step } k. \]
Main result

$C_k = \text{event that } G_k(n, r, s, m) \text{ is connected at step } k,$

$D_k = \overline{C_k} = \text{the event that } G_k(n, r, s, m) \text{ is disconnected at step } k,$

$L_k(C) = \text{the number of consecutive steps that } G(n, r(n)) \text{ remains connected starting at step } k,$

$L_k(D) = \text{the number of consecutive steps that } G(n, r(n)) \text{ remains disconnected starting at step } k.$

$$
\mathbb{E} [L_k(C) \mid D_{k-1} \land C_k] \sim \begin{cases} 
\infty & \text{if } s = \text{very small}, \\
\Theta(1) & \text{if } s = \text{mid small}, \\
1\text{step} & \text{if } s = \text{could wrap around torus},
\end{cases}
$$
Formally:

$$E[L_k(C) \mid D_{k-1} \land C_k] \sim \begin{cases} \frac{\pi}{4srn} & \text{if } srn = O(1), \\ \frac{1}{(1 - E^{4srn/\pi})} & \text{if } srn = \Theta(1), \\ 1 & \text{if } srn = \Omega(1), \end{cases}$$

Similar expression for $E[L_k(D) \mid C_{k-1} \land D_k]$
Glimpse of proof

The initial uniform distribution stays invariant as $t$ evolves: At any fixed step $k \in \mathbb{Z}$, the vertices are distributed over the torus $[0, 1)^2$ independently and u.a.r. Consequently for any $k \in \mathbb{Z}$, $G_k(n, r, s, m)$ has the same distribution as $G(n, r(n))$. 
Glimpse of proof

The initial uniform distribution stays invariant as $t$ evolves: At any fixed step $k \in \mathbb{Z}$, the vertices are distributed over the torus $[0, 1)^2$ independently and u.a.r. Consequently for any $k \in \mathbb{Z}$, $G_k(n, r, s, m)$ has the same distribution as $G(n, r(n))$. The sequence of connected/disconnected states of $G_k(n, r, s, m)$ is not Markovian, since staying connected for a long period of time makes it more likely to remain connected for one more step.
Glimpse of proof

The initial uniform distribution stays invariant as $t$ evolves: At any fixed step $k \in \mathbb{Z}$, the vertices are distributed over the torus $[0, 1)^2$ independently and u.a.r. Consequently for any $k \in \mathbb{Z}$, $G_k(n, r, s, m)$ has the same distribution as $G(n, r(n))$. The sequence of connected/disconnected states of $G_k(n, r, s, m)$ is not Markovian, since staying connected for a long period of time makes it more likely to remain connected for one more step. The main ingredient of the proof is the fact that $\Pr[C]$ and $\Pr[D]$ can be expressed in terms of the probabilities of events involving only two consecutive steps.

\[
\mathbb{E}[L_k(C) \mid D_{k-1} \land C_k] = \frac{\Pr[C_k]}{\Pr[D_{k-1} \land C_k]}
\]

\[
\mathbb{E}[L_k(D) \mid C_{k-1} \land D_k] = \frac{\Pr[D_k]}{\Pr[C_{k-1} \land D_k]}
\]
Characterization of $G_k(n, r, s, m)$ at consecutive steps $k$ and $k + 1$
Characterization of $G_k(n, r, s, m)$ at consecutive steps $k$ and $k + 1$
Open problems for dynamic random geometric graphs.

1. The unit square $I = [0, 1]^2$.
2. How is the behavior with different values of $r$, among the same set of nodes?
3. How about moving $m$ consecutive steps in the same direction before all them synchronously change direction?
4. Waypoint direction and existence of obstacles.
5. Could these models could help to develop new, more efficient, decentralized protocols?
The walkers problem: A particular case of random geometric graphs.

**GOAL:** Study the connectivity of the ad-hoc network established between the agents randomly scattered in the streets of a city (as a toroidal grid), when they move randomly through the streets:
The walkers problem on the toroidal grid.

*D., Pérez, Serna, Wormald. (2008)*

Given a set $W$ of $w$ walkers, which at each step can move N/S/E/W on the edges of a toroidal grid $T_N$ with $N = n^2$ vertices. Two walkers have direct communication if their distance is $\leq d$. We wish to study the *evolution of connectivity* in $G_t[W]$ as the walkers move.
Example.
Example.

t=0, d=3
Example.
Example.

$t=1, d=3$
Example.

t=2, d=3
Example.

Configuration

\[ a = \{(3,3),(3,5),(4,3),(6,4),(8,3)\} \]
Dynamic setting.

- The system always reaches a state representing a connected component, within finite expected time.
- The initial uniform distribution stays invariant as $t$ evolves, therefore we need to consider only the case $\mu = \Theta(1)$. 
Main Results

Interesting case $d = o(n)$,
Let $\mu = N(1 - e^{-w/N})e^{\pi d^2 w/N}$.
Expected value of the distribution of isolated components
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Interesting case $d = o(n)$,

Let $\mu = N(1 - e^{-w/N})e^{\pi d^2 w/N}$.

Expected value of the distribution of isolated components

Compute the average periods of connectivity and non-connectivity $L(C)$ and $L_k(D)$, of $G[W]$

$$\mathbb{E}[L(C)] \sim \begin{cases} \frac{1}{\mu w/N} & \text{if } dw/N = o(1), \\ \frac{1}{(1-e^{\mu(1-e^{-w/N})})} & \text{if } dw/N = \Theta(1), \\ \frac{1}{(1-e^{-\mu})}1 & \text{if } dw/N = (1), \end{cases}$$
Same result for the tree, the cycle and the $r$ dimensional grid.
Same result for the tree, the cycle and the $r$ dimensional grid.

Open problem: Extend the walkers problem to the $r$-dimensional cube, for $d = \omega(1)$. 
Thank you.