The Hiring Problem

Conrado Martínez
U. Politècnica Catalunya
Joint work with M. Archibald

July 2010
Univ. Cape Town, South Africa

The hiring problem

- The hiring problem is a simple model of decision-making under uncertainty
- It is closely related to the well-known Secretary Problem:

> *"A sequence of n candidates is to be interviewed to fill a post. For each interviewed candidate we only learn about his/her relative rank among the candidates we've seen so far. After each interview, hire and finish, or discard and interview a new candidate. The nth candidate must be hired if we have reached that far.*
> *The goal: devise an strategy that maximizes the probability of hiring the best of the n candidates."*

The hiring problem

- Originally introduced by Broder *et al.* (SODA 2008)
- The candidates are modellized by a (potentially infinite) sequence of i.i.d. random variables Q_i uniformly distributed in $[0, 1]$
- At step i you either hire or discard candidate i with score Q_i
- Decisions are irrevocable
- Goals: hire candidates at some reasonable rate, improve the "mean" quality of the company's staff

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' =$$
$$S =$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 1$$
$$S = 1$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 21$$
$$S = 11$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62{\color{red}8}17435$$
$$\sigma' = 213$$
$$S = 11{\color{red}3}$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 3241$$
$$S = 1131$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 32514$$
$$S = 11314$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 426153$$
$$S = 113143$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' = 5271643$$
$$S = 1131433$$

The hiring problem

- Our model: a permutation σ of length n, candidate i has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i+1$, s_i is the rank of the ith candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 6281743\textcolor{red}{5}$$

$$\sigma' = 62817435$$

$$S = 1131433\textcolor{red}{5}$$

Rank-based hiring

A hiring strategy is rank-based if and only if it only depends on the relative rank of the current candidate compared to the candidates seen so far.

Rank-based hiring

- Rank-based strategies modelize actual restrictions to measure qualities
- Many natural strategies are rank-based, e.g.,
  - above the best
  - above the mth best
  - above the median
  - above the P% best
- Assume only relative ranks of candidates are known, like the standard secretary problem
- Some hiring strategies are not rank-based, e.g., above the average, above a threshold.

Intermezzo: A crash course on generating functions and the symbolic method

– Excerpts from my short course "Analytic Combinatorics: A Primer"

Two basic counting principles

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets.

The Addition Principle

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint then

$$|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$$

The Multiplication Principle

$$|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$$

Combinatorial classes

Definition

A combinatorial class is a pair $(\mathcal{A}, |\cdot|)$, where $\mathcal{A}$ is a finite or denumerable set of values (combinatorial objects, combinatorial structures), $|\cdot| : \mathcal{A} \to \mathbb{N}$ is the size function and for all $n \geq 0$

$$\mathcal{A}_n = \{x \in \mathcal{A} \,|\, |x| = n\} \quad \text{is finite}$$

Combinatorial classes

Example

- $\mathcal{A}$ = all finite strings from a binary alphabet;
  $|s|$ = the length of string s
- $\mathcal{B}$ = the set of all permutations;
  $|\sigma|$ = the order of the permutation σ
- $\mathcal{C}_n$ = the partitions of the integer n; $|p| = n$ if $p \in \mathcal{C}_n$

Labelled and unlabelled classes

- In unlabelled classes, objects are made up of indistinguisable atoms; an atom is an object of size 1
- In labelled classes, objects are made up of distinguishable atoms; in an object of size n, each of its n atoms bears a distinct label from $\{1, \ldots, n\}$

Counting generating functions

Definition

Let $a_n = \#\mathcal{A}_n =$ the number of objects of size n in $\mathcal{A}$. Then the formal power series

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}$$

is the (ordinary) generating function of the class $\mathcal{A}$.
The coefficient of z^n in $A(z)$ is denoted $[z^n]A(z)$:

$$[z^n]A(z) = [z^n] \sum_{n \geq 0} a_n z^n = a_n$$

Counting generating functions

Ordinary generating functions (OGFs) are mostly used to enumerate unlabelled classes.

Example

$$
\begin{aligned}
\mathcal{L} &= \{w \in (0+1)^* \mid w \text{ does not contain two consecutive 0's}\} \\
&= \{\epsilon, 0, 1, 01, 10, 11, 010, 011, 101, 110, 111, \ldots\} \\
L(z) &= z^{|\epsilon|} + z^{|0|} + z^{|1|} + z^{|01|} + z^{|10|} + z^{|11|} + \cdots \\
&= 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \cdots
\end{aligned}
$$

Exercise: Can you guess the value of $L_n = [z^n]L(z)$?

Counting generating functions

Definition

Let $a_n = \#\mathcal{A}_n =$ the number of objects of size n in $\mathcal{A}$. Then the formal power series

$$\hat{A}(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!} = \sum_{\alpha\in\mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!}$$

is the exponential generating function of the class $\mathcal{A}$.

Counting generating functions

Exponential generating functions (EGFs) are used to enumerate labelled classes.

Example

$$
\begin{aligned}
\mathcal{C} &= \text{circular permutations} \\
&= \{\epsilon, 1, 12, 123, 132, 1234, 1243, 1324, 1342, \\
&\qquad 1423, 1432, 12345, \ldots\} \\
\hat{C}(z) &= \frac{1}{0!} + \frac{z}{1!} + \frac{z^2}{2!} + 2\frac{z^3}{3!} + 6\frac{z^4}{4!} + \cdots \\
c_n &= n! \cdot [z^n]\hat{C}(z) = (n-1)!, \qquad n > 0
\end{aligned}
$$

Disjoint union

Let $\mathcal{C} = \mathcal{A} + \mathcal{B}$, the disjoint union of the unlabelled classes $\mathcal{A}$ and $\mathcal{B}$ ($\mathcal{A} \cap \mathcal{B} = \emptyset$). Then

$$C(z) = A(z) + B(z)$$

And

$$c_n = [z^n]C(z) = [z^n]A(z) + [z^n]B(z) = a_n + b_n$$

Cartesian product

Let $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, the Cartesian product of the unlabelled classes $\mathcal{A}$ and $\mathcal{B}$. The size of $(\alpha, \beta) \in \mathcal{C}$, where $a \in \mathcal{A}$ and $\beta \in \mathcal{B}$, is the sum of sizes: $|(\alpha, \beta)| = |\alpha| + |\beta|$.
Then

$$C(z) = A(z) \cdot B(z)$$

Proof.

$$C(z) = \sum_{\gamma \in \mathcal{C}} z^{|\gamma|} = \sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} z^{|\alpha|+|\beta|} = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} z^{|\alpha|} \cdot z^{|\beta|}$$

$$= \left(\sum_{\alpha \in \mathcal{A}} z^{|\alpha|} \right) \cdot \left(\sum_{\beta \in \mathcal{B}} z^{|\beta|} \right) = A(z) \cdot B(z)$$

$\square$

Cartesian product

The nth coefficient of the OGF for a Cartesian product is the *convolution* of the coefficients $\{a_n\}$ and $\{b_n\}$:

$$\begin{aligned} c_n &= [z^n]C(z) = [z^n]A(z) \cdot B(z) \\ &= \sum_{k=0}^{n} a_k \, b_{n-k} \end{aligned}$$

Sequences

Let $\mathcal{A}$ be a class without any empty object ($\mathcal{A}_0 = \emptyset$). The class $\mathcal{C} = \mathrm{Seq}(\mathcal{A})$ denotes the class of sequences of $\mathcal{A}$'s.

$$\begin{aligned}\mathcal{C} &= \{(\alpha_1, \ldots, \alpha_k) \,|\, k \geq 0, \alpha_i \in \mathcal{A}\} \\ &= \{\epsilon\} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \cdots = \{\epsilon\} + \mathcal{A} \times \mathcal{C}\end{aligned}$$

Then

$$C(z) = \frac{1}{1 - A(z)}$$

Proof.

$$C(z) = 1 + A(z) + A^2(z) + A^3(z) + \cdots = 1 + A(z) \cdot C(z)$$

□

Labelled objects

Disjoint unions of labelled classes are defined as for unlabelled classes and $\hat{C}(z) = \hat{A}(z) + \hat{B}(z)$, for $\mathcal{C} = \mathcal{A} + \mathcal{B}$. Also, $c_n = a_n + b_n$.

To define labelled products, we must take into account that for each pair (α, β) where $|\alpha| = k$ and $|\alpha| + |\beta| = n$, we construct $\binom{n}{k}$ distinct pairs by consistently relabelling the atoms of α and β:

$$\begin{aligned}
\alpha = (2,1,4,3), &\quad \beta = (1,3,2) \\
\alpha \times \beta = \{&(2,1,4,3,5,7,6),(2,1,5,3,4,7,6),\ldots, \\
&(5,4,7,6,1,3,2)\}
\end{aligned}$$

$$\#(\alpha \times \beta) = \binom{7}{4} = 35$$

The size of an element in $\alpha \times \beta$ is $|\alpha| + |\beta|$.

Labelled products

For a class $\mathcal{C}$ that is labelled product of two labelled classes $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \bigcup_{\substack{\alpha \in \mathcal{A} \\ \beta \in \mathcal{B}}} \alpha \times \beta$$

the following relation holds for the corresponding EGFs

$$\begin{aligned}
\hat{C}(z) &= \sum_{\gamma \in \mathcal{C}} \frac{z^{|\gamma|!}}{|\gamma|!} = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} \binom{|\alpha| + |\beta|}{|\alpha|} \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} \\
&= \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} \frac{1}{|\alpha|!|\beta|!} z^{|\alpha|+|\beta|} = \left(\sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} \right) \cdot \left(\sum_{\beta \in \mathcal{B}} \frac{z^{|\beta|}}{|\beta|!} \right) \\
&= \hat{A}(z) \cdot \hat{B}(z)
\end{aligned}$$

Labelled products

The nth coefficient of $\hat{C}(z) = \hat{A}(z) \cdot \hat{B}(z)$ is also a convolution

$$c_n = [z^n]\hat{C}(z) = \sum_{k=0}^{n} \binom{n}{k} a_k \, b_{n-k}$$

Sequences

Sequences of labelled object are defined as in the case of unlabelled objects. The construction $\mathcal{C} = \text{Seq}(\mathcal{A})$ is well defined if $\mathcal{A}_0 = \emptyset$.
If $\mathcal{C} = \text{Seq}(\mathcal{A}) = \{\epsilon\} + \mathcal{A} \times \mathcal{C}$ then

$$\hat{C}(z) = \frac{1}{1 - \hat{A}(z)}$$

Example

Permutations are labelled sequences of atoms, $\mathcal{P} = \text{Seq}(\mathcal{Z})$. Hence,

$$\hat{P}(z) = \frac{1}{1 - z} = \sum_{n \geq 0} z^n$$

$$n! \cdot [z^n]\hat{P}(z) = n!$$

A dictionary of admissible unlabelled operators

Class	OGF	Name
ϵ	1	Epsilon
$\mathcal{Z}$	z	Atomic
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$	Disjoint union
$\mathcal{A}\times\mathcal{B}$	$A(z)\cdot B(z)$	Product
$\mathrm{Seq}(\mathcal{A})$	$\frac{1}{1-A(z)}$	Sequence
$\Theta\mathcal{A}$	$\Theta A(z) = zA'(z)$	Marking
$\mathrm{MSet}(\mathcal{A})$	$\exp\left(\sum_{k>0} A(z^k)/k\right)$	Multiset
$\mathrm{PSet}(\mathcal{A})$	$\exp\left(\sum_{k>0} (-1)^k A(z^k)/k\right)$	Powerset
$\mathrm{Cycle}(\mathcal{A})$	$\sum_{k>0} \frac{\phi(k)}{k} \ln \frac{1}{1-A(z^k)}$	Cycle

A dictionary of admissible labelled operators

Class	EGF	Name
ϵ	1	Epsilon
$\mathcal{Z}$	z	Atomic
$\mathcal{A} + \mathcal{B}$	$\hat{A}(z) + \hat{B}(z)$	Disjoint union
$\mathcal{A} \times \mathcal{B}$	$\hat{A}(z) \cdot \hat{B}(z)$	Product
$\mathrm{Seq}(\mathcal{A})$	$\frac{1}{1-\hat{A}(z)}$	Sequence
$\Theta\mathcal{A}$	$\Theta\hat{A}(z) = z\hat{A}'(z)$	Marking
$\mathrm{Set}(\mathcal{A})$	$\exp(\hat{A}(z))$	Set
$\mathrm{Cycle}(\mathcal{A})$	$\ln\left(\frac{1}{1-\hat{A}(z)}\right)$	Cycle

Bivariate generating functions

We need often to study some characteristic of combinatorial structures, e. g., the number of left-to-right maxima in a permutation, the height of a rooted tree, the number of complex components in a graph, etc.
Suppose $X : \mathcal{A}_n \to \mathbb{N}$ is a characteristic under study. Let

$$a_{n,k} = \#\{\alpha \in \mathcal{A} \,|\, |\alpha| = n, X(\alpha) = k\}$$

We can view the restriction $X_n : \mathcal{A}_n \to \mathbb{N}$ as a random variable.
Then under the usual uniform model

$$\mathbb{P}[X_n = k] = \frac{a_{n,k}}{a_n}$$

Bivariate generating functions

Define

$$A(z,u) = \sum_{n,k \geq 0} a_{n,k} z^n u^k$$
$$= \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} u^{X(\alpha)}$$

Then $a_{n,k} = [z^n u^k] A(z,u)$ and

$$\mathbb{P}[X_n = k] = \frac{[z^n u^k] A(z,u)}{[z^n] A(z,1)}$$

Bivariate generating functions

We can also define

$$B(z,u) = \sum_{n,k \geq 0} \mathbb{P}[X_n = k]\ z^n u^k$$
$$= \sum_{\alpha \in \mathcal{A}} \mathbb{P}[\alpha]\, z^{|\alpha|} u^{X(\alpha)}$$

and thus $B(z,u)$ is a generating function whose coefficient of z^n is the probability generating function of the r.v. X_n

$$B(z,u) = \sum_{n \geq 0} P_n(u) z^n$$
$$P_n(u) = [z^n]B(z,u) = \sum_{k \geq 0} \mathbb{P}[X_n = k]\, u^k$$

Bivariate generating functions

Proposition

If $P(u)$ is the probability generating function of a random variable X then

$$\begin{aligned} P(1) &= 1, \\ P'(1) &= \mathbb{E}[X], \\ P''(1) &= \mathbb{E}\left[X^{\underline{2}}\right] = \mathbb{E}[X(X-1)], \\ \mathbb{V}[X] &= P''(1) + P'(1) - (P'(1))^2 \end{aligned}$$

Bivariate generating functions

We can study the moments of X_n by successive differentiation of $B(z,u)$ (or $A(z,u)$). For instance,

$$\overline{B}(z) = \sum_{n\geq 0} \mathbb{E}[X_n]\, z^n = \left.\frac{\partial B}{\partial u}\right|_{u=1}$$

For the rth factorial moments of X_n

$$B^{(r)}(z) = \sum_{n\geq 0} \mathbb{E}[X_n{}^{\underline{r}}]\, z^n = \left.\frac{\partial^r B}{\partial u^r}\right|_{u=1}$$

$$X_n{}^{\underline{r}} = X_n(X_n - 1)\cdot\dots\cdot(X_n - r + 1)$$

The number of left-to-right maxima in a permutation

Consider the following specification for permutations

$$\mathcal{P} = \{\emptyset\} + \mathcal{P} \times Z$$

The BGF for the probability that a random permutation of size n has k left-to-right maxima is

$$M(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{X(\sigma)},$$

where $X(\sigma) = \#$ of left-to-right maxima in σ

The number of left-to-right maxima in a permutation

With the recursive descomposition of permutations and since the last element of a permutation of size n is a left-to-right maxima iff its label is n

$$M(z,u) = \sum_{\sigma \in \mathcal{P}} \sum_{1 \leq j \leq |\sigma|+1} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{X(\sigma) + [\![j = |\sigma|+1]\!]}$$

$[\![P]\!] = 1$ if P is true, $[\![P]\!] = 0$ otherwise.

The number of left-to-right maxima in a permutation

$$M(z,u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{X(\sigma)} \sum_{1 \leq j \leq |\sigma|+1} u^{[\![j=|\sigma|+1]\!]}$$

$$= \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{X\sigma)} (|\sigma| + u)$$

Taking derivatives w.r.t. z

$$\frac{\partial}{\partial z} M = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{X\sigma)} (|\sigma| + u) = z \frac{\partial}{\partial z} M + uM$$

Hence,

$$(1-z)\frac{\partial}{\partial z} M(z,u) - uM(z,u) = 0$$

The number of left-to-right maxima in a permutation

Solving, since $M(0,u) = 1$

$$M(z,u) = \left(\frac{1}{1-z}\right)^u = \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} u^k$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the (signless) Stirling numbers of the first kind, also called Stirling cycle numbers.

Taking the derivative w.r.t. u and setting $u = 1$

$$m(z) = \frac{\partial}{\partial z} M(z,u)\bigg|_{u=1} = \frac{1}{1-z} \ln \frac{1}{1-z}$$

Thus the average number of left-to-right maxima in a random permutation of size n is

$$[z^n]m(z) = \mathbb{E}[X_n] = H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + \gamma + O(1/n)$$

$$\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{\ell} z^\ell \sum_{m>0} \frac{z^m}{m} = \sum_{n\geq 0} z^n \sum_{k=1}^{n} \frac{1}{k}$$

. . . back to bussiness now!

Rank-based hiring

- The recursive decomposition of permutations

$$\mathcal{P} = \epsilon + \mathcal{P} \times Z$$

  is the natural choice for the analysis of rank-based strategies, with $\times$ denoting the labelled product.

- For each σ in $\mathcal{P}$, $\{\sigma\} \times Z$ is the set of $|\sigma| + 1$ permutations

$$\{\sigma \star 1, \sigma \star 2, \ldots, \sigma \star (n+1)\}, \qquad n = |\sigma|$$

  $\sigma \star j$ denotes the permutation one gets after relabelling j, $j+1$, ..., $n = |\sigma|$ in σ to $j+1$, $j+2$, ..., $n+1$ and appending j at the end

Example

$$32451 \star 3 = 425613$$

$$32451 \star 2 = 435612$$

Rank-based hiring

- $\mathcal{H}(\sigma)$ = the set of candidates hired in permutation σ
- $h(\sigma) = \#\mathcal{H}(\sigma)$
- Let $X_j(\sigma) = 1$ if candidate with score j is hired after σ and $X_j(\sigma) = 0$ otherwise.
- $h(\sigma \star j) = h(\sigma) + X_j(\sigma)$

Rank-based hiring

Theorem

Let $H(z,u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)}$.
Then

$$(1-z)\frac{\partial}{\partial z}H(z,u) - H(z,u) = (u-1)\sum_{\sigma \in \mathcal{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}u^{h(\sigma)},$$

where $X(\sigma)$ *the number of* j *such that* $X_j(\sigma) = 1$.

Rank-based hiring

We can write $h(\sigma) = 0$ if σ is the empty permutation and $h(\sigma \star j) = h(\sigma) + X_j(\sigma)$.

$$
\begin{aligned}
H(z,u) &= \sum_{\sigma\in\mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma\in\mathcal{P}_n} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} \\
&= 1 + \sum_{n>0} \sum_{1\le j\le n} \sum_{\sigma\in\mathcal{P}_{n-1}} \frac{z^{|\sigma\star j|}}{|\sigma\star j|!} u^{h(\sigma\star j)} \\
&= 1 + \sum_{n>0} \sum_{1\le j\le n} \sum_{\sigma\in\mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{h(\sigma)+X_j(\sigma)} \\
&= 1 + \sum_{n>0} \sum_{\sigma\in\mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{h(\sigma)} \sum_{1\le j\le n} u^{X_j(\sigma)}.
\end{aligned}
$$

Rank-based hiring

Since $X_j(\sigma)$ is either 0 or 1 for all j and all σ, we have

$$\sum_{1 \leq j \leq n} u^{X_j(\sigma)} = (|\sigma| + 1 - X(\sigma)) + uX(\sigma),$$

where $X(\sigma) = \sum_{1 \leq j \leq |\sigma|+1} X_j(\sigma)$.

$$H(z, u) = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma| + 1)!} u^{h(\sigma)} \Big((|\sigma| + 1 - X(\sigma)) + uX(\sigma) \Big).$$

The theorem follows after differentiation and a few additional algebraic manipulations.

Pragmatic strategies

A hiring strategy is pragmatic if and only if

- Whenever it would hire a candidate with score j, it would hire a candidate with a larger score

$$X_j(\sigma) = 1 \implies X_{j'}(\sigma) = 1 \qquad \text{for all } j' \geq j$$

- The number of scores it would potentially hire increases at most by one if and only if the candidate in the previous step was hired

$$X(\sigma \star j) \leq X(\sigma) + X_j(\sigma)$$

Pragmatic strategies

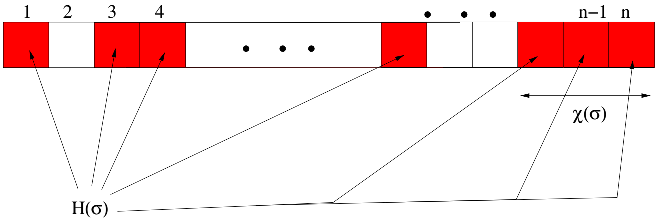
- The first condition is very natural and reasonable; the second one is technically necessary for several results we discuss later
- Above the best, above the mth best, above the $P\%$ best, ... are all pragmatic

Pragmatic strategies

Theorem

For any pragmatic hiring strategy and any permutation σ, the $X(\sigma)$ best candidates of σ have been hired (and possibly others).

Pragmatic strategies



Pragmatic strategies

Let r_n denote the rank of the last hired candidate in a random permutation, and

$$g_n = 1 - \frac{r_n}{n}$$

is called the gap.

Theorem

For any pragmatic hiring strategy,

$$\mathbb{E}[g_n] = \frac{1}{2n}(\mathbb{E}[X_n] - 1),$$

where $\mathbb{E}[X_n] = [z^n] \sum_{\sigma \in \mathcal{P}} X(\sigma) z^{|\sigma|} / |\sigma|!$.

Hiring above the maximum

Candidate i is hired if and only if her score is above the score of the best currently hired candidate.

- $X(\sigma) = 1$
- $\mathcal{H}(\sigma) = \{i : i \text{ is a left-to-right maximum}\}$
- $\mathbb{E}[h_n] = [z^n] \left.\frac{\partial H}{\partial u}\right|_{u=1} = \ln n + O(1)$
- Variance of h_n is also $\ln n + O(1)$ and after proper normalization h_n^* converges to $\mathcal{N}(0,1)$

Hiring above the mth best

Candidate i is hired if and only if her score is above the score of the mth best currently hired candidate.

- $X(\sigma) = |\sigma| + 1$ if $|\sigma| < m$; $X(\sigma) = m$ if $|\sigma| \geq m$
- $\mathbb{E}[h_n] = [z^n] \left.\frac{\partial H}{\partial u}\right|_{u=1} = m \ln n + O(1)$ for fixed m
- Variance of h_n is also $m \ln n + O(1)$ and after proper normalization h_n^* converges to $\mathcal{N}(0, 1)$
- The case of arbitrary m can be studied by introducing $\mathrm{H}(z, u, v) = \sum_{m \geq 1} v^m H^{(m)}(z, u)$, where $H^{(m)}(z, u)$ is the GF that corresponds to a given particular m.
- We can show that $\mathbb{E}[h_n] = m(H_n - H_m + 1) \sim m \ln(n/m) + m + O(1)$, with H_n the nth harmonic number

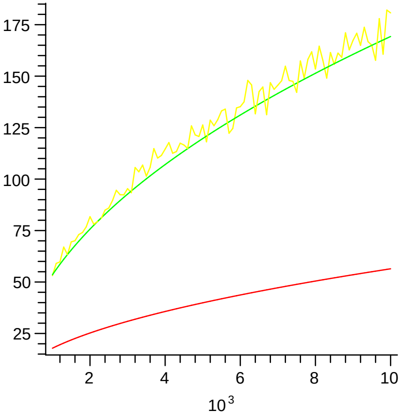
Hiring above the median

Candidate i is hired if and only if her score is above the score of the median of the scores of currently hired candidates.

- $X(\sigma) = \lceil (h(\sigma) + 1)/2 \rceil$
- $\sqrt{\frac{n}{\pi}}(1 + O(n^{-1})) \leq \mathbb{E}[h_n] \leq 3\sqrt{\frac{n}{\pi}}(1 + O(n^{-1}))$
- This result follows easily by using previous theorem with $X_L(\sigma) = (h(\sigma) + 1)/2$ and $X_U(\sigma) = (h(\sigma) + 3)/2$ to lower and upper bound

Hiring above the median

$n \in \{1000, \ldots, 10000\}$, $M = 100$ random permutations for each n



In red: lower bound (using X_L); in green: upper bound (using X_U); in yellow: simulation

Final remarks

- Other quantities, e.g. time of the last hiring, etc. can also be analyzed using techniques from analytic combinatorics
- We have also analyzed hiring above the $P\%$ best candidate with the same machinery, actually we have explicit solutions for $H(z, u)$
- We have extensions of these results to cope with randomized hiring strategies
- Many variants of the problem are interesting and natural; for instance, include firing policies

Thanks for your attention!