Analysis of Approximate Quickselect and Related Problems

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Introduction

- Quickselect finds the kth smallest element out of n given elements with average cost $\Theta(n)$
- Approximate Quickselect selects, out of n given elements, an element whose rank k fails within a prespecified rank $[i..j]$
- We analyze the exact number of passes (recursive calls) and number of key comparisons made by AQS, for arbitrary i, j and n
- Asymptotic estimates follow easily from the exact results

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  1. The number of moves in which the ith element gets involved when selecting the jth smallest element out of n
  2. The number of common ancestors of the nodes of rank i and j in a random binary search tree (BST) of size n
  3. The size of the smallest subtree containing the nodes i and j in a random BST of size n
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The algorithm

Ensure: Array $A[l..r]$, integers i and j with $l \leq i \leq j \leq r$
Require: Returns a value k, with $i \leq k \leq j$, $A[k]$ has rank between $i - l + 1$ and $j - l + 1$ in the array $A[l..r]$

procedure AQS(A, i, j, l, r)
  if $r - l \leq j - i$ **then return** l
  end if
  PARTITION(A, l, r, k)
  $\{ \forall m : (l \leq m < k) \Rightarrow A[m] \leq A[k]$, and
    $\forall m : (k < m \leq r) \Rightarrow A[k] \leq A[m] \}$
  if $j < k$ **then return** AQS($A, i, j, l, k - 1$)
  else if $i > k$ **then return** AQS($A, i, j, k + 1, r$)
  else return k
  end if
end procedure

The number of passes

$P_{n,i,j}$ = the average number of recursive calls to AQS to select an element of rank $k \in [i..j]$ out of n

$$P_{n,i,j} = 1 + \frac{1}{n}\sum_{k=1}^{i-1} P_{n-k,i-k,j-k} + \frac{1}{n}\sum_{k=j+1}^{n} P_{k-1,i,j}, \quad \text{for } 1 \leq i \leq j \leq n$$

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The initial recursive call

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If the pivot lands at $k < i$ we continue in the right subarray of $n - k$ elements looking for an element with rank in $[i - k..j - k]$

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If the pivot lands at $k > j$ we continue in the left subarray of $k-1$ elements looking for an element with rank in $[i..j]$

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If the pivots lands at k, $i \leq k \leq j$, we are done

The number of comparisons

$C_{n,i,j}$ = the average number of key comparisons in AQS to select an element of rank $k \in [i..j]$ out of n

$$C_{n,i,j} = n-1 + \frac{1}{n} \sum_{k=1}^{i-1} C_{n-k,i-k,j-k} + \frac{1}{n} \sum_{k=j+1}^{n} C_{k-1,i,j}, \quad \text{for } 1 \le i \le j \le n$$

The generic trivariate recurrence

$T_{n,i,j}$ = generic "toll" function

$$X_{n,i,j} = T_{n,i,j} + \frac{1}{n} \sum_{k=1}^{i-1} X_{n-k,i-k,j-k} + \frac{1}{n} \sum_{k=j+1}^{n} X_{k-1,i,j}, \quad \text{for } 1 \le i \le j \le n$$

Example

- $T_{n,i,j} = 1 \Rightarrow$ passes
- $T_{n,i,j} = n - 1 \Rightarrow$ comparisons
- $T_{n,i,j} = \frac{n}{6} + O(1) \Rightarrow$ swaps
- $i = j \Rightarrow$ Quickselect

The generic trivariate recurrence

- Define:

$$X(z,u_1,u_2) := \sum_{i\geq 1}\sum_{j\geq i}\sum_{n\geq j} X_{n,i,j} z^n u_1^i u_2^j,$$

$$T(z,u_1,u_2) := \sum_{i\geq 1}\sum_{j\geq i}\sum_{n\geq j} T_{n,i,j} z^n u_1^i u_2^j.$$

- The trivariate recurrence translates to

$$\frac{\partial}{\partial z} X(z,u_1,u_2) = \left(\frac{1}{1-z} + \frac{u_1 u_2}{1-zu_1u_2}\right) X(z,u_1,u_2) + \frac{\partial}{\partial z} T(z,u_1,u$$

with initial condition $X(0,u_1,u_2) = 0$.

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Lemma

$$X(z,u_1,u_2) = \frac{1}{(1-z)(1-zu_1u_2)} \times \int_0^z (1-t)(1-u_1u_2t)\Big(\frac{\partial}{\partial t}T(t,u_1,u_2)\Big)dt.$$

The generic trivariate recurrence

Theorem

$$X_{n,i,j} = \sum_{\ell=1}^{i-1} \sum_{k=j-i+\ell}^{n-i+\ell-1} \frac{2T_{k,\ell,j-i+\ell}}{(k+1)(k+2)} + \sum_{\ell=1}^{i-1} \frac{T_{n-i+\ell,\ell,j-i+\ell}}{n-i+\ell+1} + \sum_{k=j}^{n-1} \frac{T_{k,i,j}}{k+1} + T_{n,i,j}$$

Setting $i = j$ we rederive the generic solution for Quickselect-like recurrences by Kuba (2006).

Analyzing AQS

Theorem

The expected number of passes in AQS is*

$$P_{n,i,j} = H_j + H_{n-i+1} - 2H_{j-i+1} + 1.$$

The expected number of comparisons in AQS is

$$\begin{aligned} C_{n,i,j} = 2(n+1)H_n + 2(j-i+4)H_{j-i+1} - 2(j+2)H_j \\ - 2(n-i+3)H_{n-i+1} + 2n - j + i - 2. \end{aligned}$$

$$(^*) \quad H_n := \sum_{1 \le k \le n} \frac{1}{k}, \quad n\text{th harmonic number}$$

Moves in Quickselect

Consider the ith smallest element in the array $A[1..n]$. How many times do we move it around when selecting the jth smallest element in A?

Moves in Quickselect

We make a case analysis, comparing with the position k where the pivot lands after a partitioning step:

- if $k < i \leq j$ or $k < j \leq i$, Quickselect continues recursively in $A[k + 1..n]$ that does contain ith element
- if $j \leq i < k$ or $i \leq j < k$, Quickselect continues in $A[1..k - 1]$ that does contain the ith element
- if $i \leq k \leq j$ or $j \leq k \leq i$, either Quickselect stops or continues in a subarray not containing ith element

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Moves in Quickselect

To find the toll function (number of moves where i participates) in a single partitioning step, we also consider three cases:

1. $i = k \Rightarrow$ the element i is moved (once)
2. $i < k \Rightarrow$ the element i is moved if it were in $A[k..n] \Rightarrow$ prob $= (n - k + 1)/(n - 1)$
3. $i > k \Rightarrow$ the element is moved if it were in $A[2..k] \Rightarrow$ prob $= (k - 1)/(n - 1)$

Moves in Quickselect

$M_{n,i,j}$:= The expected number of moves of the ith element when selecting the jth smallest out of n

$$M_{n,i,j} = \frac{1}{n}\sum_{k=1}^{i-1} M_{n-k,i-k,j-k} + \frac{1}{n}\sum_{k=j+1}^{n} M_{k-1,i,j} + \frac{(i-1)(i-2)}{2n(n-1)} + \frac{(n-i)(n-i+1)}{2n(n-1)} + \frac{1}{n}, \quad 1 \leq i < j \leq n$$

Analogous recurrences for $i = j$ and $j < i$, all three solved using the theorem for trivariate recurrences

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Binary search trees

- $A_{n,i,j}$:= average # of common ancestors of nodes i and j
- $S_{n,i,j}$:= average size of smallest subtree containing nodes i and j
- $D_{n,i,j}$:= average distance (# of edges) between nodes i and j

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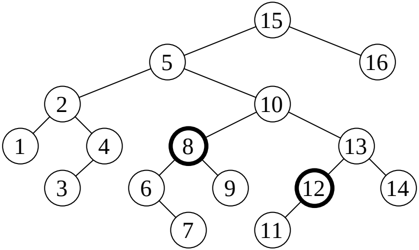

Example

$$A_{16,8,12} = 3 \qquad \text{(nodes 15, 5, 10)}$$
$$S_{16,8,12} = 9 \qquad \text{(subtree rooted at 10)}$$
$$D_{16,8,12} = 3$$

Binary search trees

if $T = \circ(L, R)$ is random binary search tree (BST) of size $n > 0$, then

1. Any element has identical probability of being the root, thus

$$\Pr\{|L| = k - 1 \,|\, |T| = n\} = \frac{1}{n}, \qquad 1 \le k \le n$$

2. L and R are independent random BSTs of sizes $k - 1$ and $n - k$, $1 \le k \le n$

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Common ancestors

Suppose that the root of the BST is the kth element.

- if $i \leq j < k$ the number of common ancestors is 1 + the number of common ancestors in a random of BST of size $k - 1$
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- if $i \leq k \leq j$, then there is only one common ancestor (k)

Lemma

$$A_{n,i,j} = P_{n,i,j} = passes\ in\ AQS$$

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Size of subtree of LCA

The recurrence for $S_{n,i,j}$ follows the usual pattern:

$$\begin{aligned} S_{n,i,j} &= \frac{1}{n}\sum_{k=1}^{i-1} S_{n-k,i-k,j-k} + \frac{1}{n}\sum_{k=j+1}^{n} S_{k-1,i,j} + \frac{1}{n}\sum_{k=i}^{j} n \\ &= \frac{1}{n}\sum_{k=1}^{i-1} S_{n-k,i-k,j-k} + \frac{1}{n}\sum_{k=j+1}^{n} S_{k-1,i,j} + j - i + 1 \end{aligned}$$

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\end{aligned}
$$

The toll function is n only when the pivot k satisfies $i \leq k \leq j$

Size of subtree of LCA

Apply the theorem for trivariate recurrences with $T_{n,i,j} := j - i + 1$

Theorem

$$
\begin{aligned}
S_{n,i,j} &= (j-i+1)\bigl(H_j + H_{n-i+1} - 2H_{j-i+1} + 1\bigr) = (j-i+1)\cdot A_{n,i,j} \\
&\approx (j-i+1)\bigl(\log j + \log(n-i+1) - 2\log(j-i+1) + 1\bigr)
\end{aligned}
$$

Distance

The recurrence for $D_{n,i,j}$:

$$D_{n,i,j} = \frac{1}{n}\sum_{k=1}^{i-1} D_{n-k,i-k,j-k} + \frac{1}{n}\sum_{k=j+1}^{n} D_{k-1,i,j} + \frac{1}{n}\sum_{k=i}^{j}(A_{k-1,i,i} + A_{n-k,j-k,j-k} + 2)$$

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If k is the LCA of i and j, the distance between i and j is the depth of i in the left subtree ($A_{k-1,i,i}$), plus the depth of j (the $(j-k)$th element) in the right subtree ($A_{n-k,j-k,j-k}$), plus 2

Distance

Since we know $A_{n,i,j}$, we can obtain the toll function for distances

$$\frac{1}{n}\sum_{k=i}^{j}(A_{k-1,i,i} + A_{n-k,j-k,j-k} + 2)$$
$$= \frac{j-i+1}{n}\left(H_i + H_{n+1-j} + 2H_{j+1-i} - 2\right)$$

Distance

The last step is to apply the theorem of trivariate recurrences with the toll function above (quite laboriously!)

Theorem

$$D_{n,i,j} = 4H_{j+1-i} - (H_j - H_i) - (H_{n+1-i} - H_{n+1-j}) - 3$$