Rank Selection in Multidimensional Data

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Univ. Politècnica Catalunya
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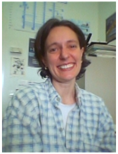
Joint work with:

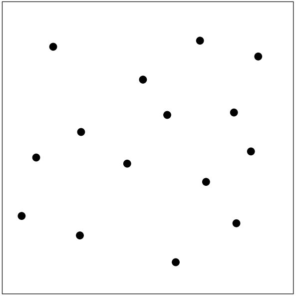


A. Duch

R.M. Jiménez

Introduction

The problem: Given a collection of n multidimensional records, each with K coordinates, and values i, $1 \leqslant i \leqslant n$, and j, $1 \leqslant j \leqslant K$, find the i-th record along the j-th coordinate



$n = 15$
$K = 2$

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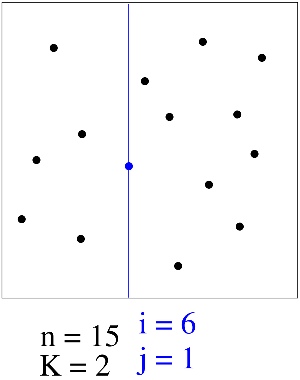

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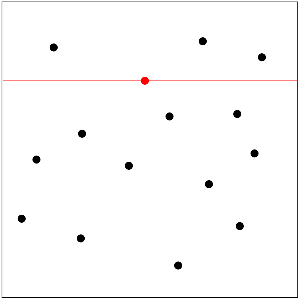

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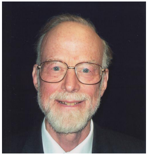

C.A.R. Hoare

R. Floyd

R. Rivest

Easy solution: use an efficient selection algorithm, with (expected) linear cost, e.g., using Hoare's or Floyd and Rivest's algorithms for selection

Introduction

- What if the collection is organized in some multidimensional index? (e.g., a K-d tree, a quadtree, ...)
- If $K = 1$ and the collection of n records is stored in some kind of binary search tree $\Rightarrow$ (expected) time $\Theta(\log n)$, using some little extra space
- We look for an algorithm that uses space $\Theta(n)$, independent of K
- The data structure for the n records should also efficiently support usual spatial queries, e.g., orthogonal range search
- We assume w.l.o.g. the n records are points from $[0, 1]^K$

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K-d trees

J.L. Bentley

Definition

A K-d tree for a set $X \subset [0,1]^K$ is either the empty tree if $X = \emptyset$ or a binary tree where:

- the root contains $y \in X$ and some value j, $1 \leqslant j \leqslant K$
- the left subtree is a K-d tree for $X^- = \{x \in X \,|\, x_j < y_j\}$
- the right subtree is a K-d tree for $X^+ = \{x \in X \,|\, y_j < x_j\}$

K-d trees

- In standard K-d trees, discriminants (the values j) of the nodes are cyclically assigned by level: the root has $j = 1$, the nodes in next level have $j = 2, \ldots,$ nodes at level K have $j = K$, then at level $K + 1$ all nodes have $j = 1$, etc.
- In relaxed K-d trees discriminants are assigned uniformly at random
- In squarish K-d trees discriminants are assigned to divide the region corresponding to each node as evenly as possible

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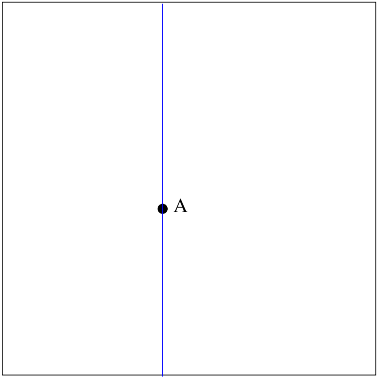

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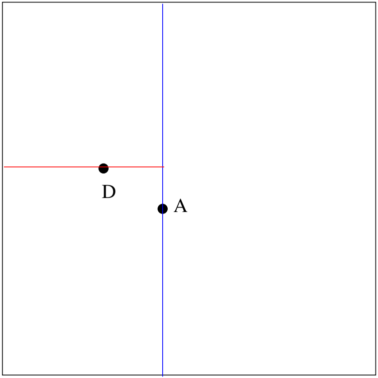

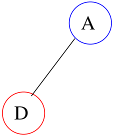

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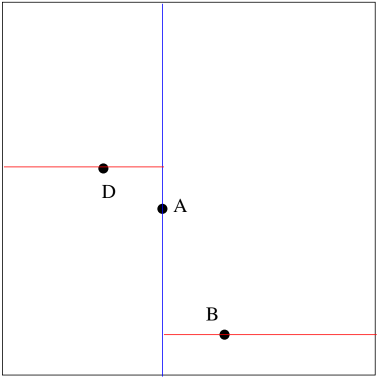

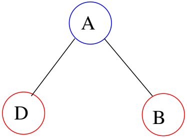

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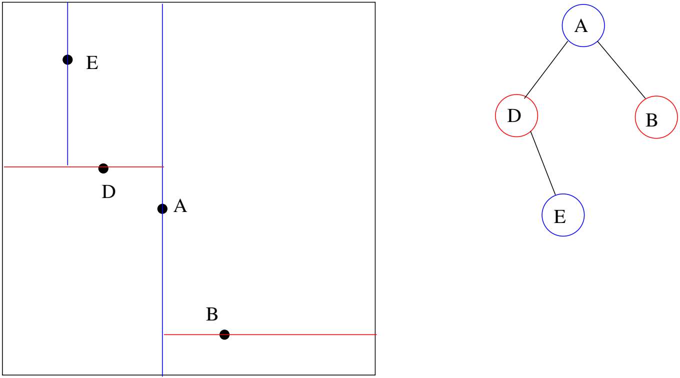

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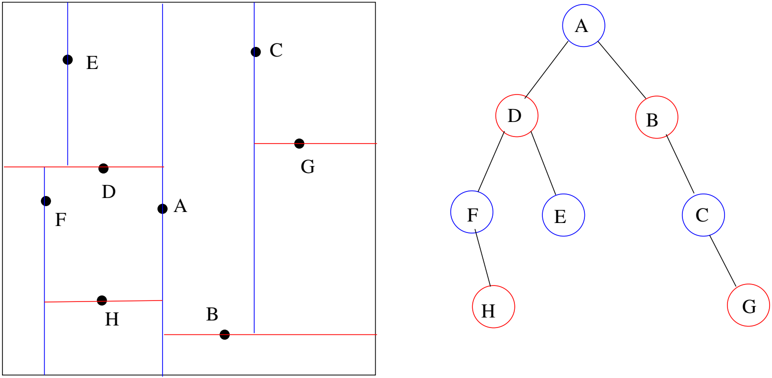

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- In a partial match query we are given a query $q = (q_1, \ldots, q_K)$ where s coordinates are specified and $K - s$ are "don't cares"
- The goal is to find all records in a collection that satisfy the query
- Flajolet and Puech (1986) showed that a partial match in a random standard K-d tree of size n has expected cost $\Theta(n^{\alpha(s/K)})$, where $\alpha(x) = 1 - x + \phi(x)$, $0 \leqslant \phi(x) < 0.07$
- Similar results have been proved for other variants of K-d trees, quadtrees, etc.

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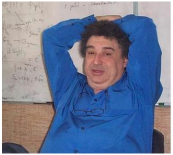

Ph. Flajolet

C. Puech

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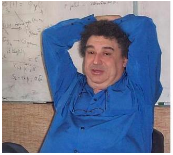

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K-d trees

L. Devroye

- Orthogonal range queries ask for all records falling inside an hyperrectangle (with sides parallel to the axis); their expected cost has been analyzed by Chanzy, Devroye and Zamora-Cura (2001) and Duch and Martínez (2002):

$$n \cdot \text{volume of query} + n^{\alpha(1/K)} \cdot \text{perimeter of query} + \text{l.o.t.}$$

The algorithm

Our algorithm has three main steps

- The main loop starts with a strip $x_j \in [low, high] = [0, 1]$ and explores the K-d tree, reducing the strip in such a way that it always contains the i-th record along coordinate j
- When the main loop finishes, it has found the sought element (if it is stored in a node that discriminates w.r.t. j) or the strip does only contain nodes discriminating w.r.t. a coordinate $\neq j$; if needed, the second step performs an orthogonal range search to locate all records within the strip
- A conventional selection algorithm is used to find the sought element among the elements reported in the previous step

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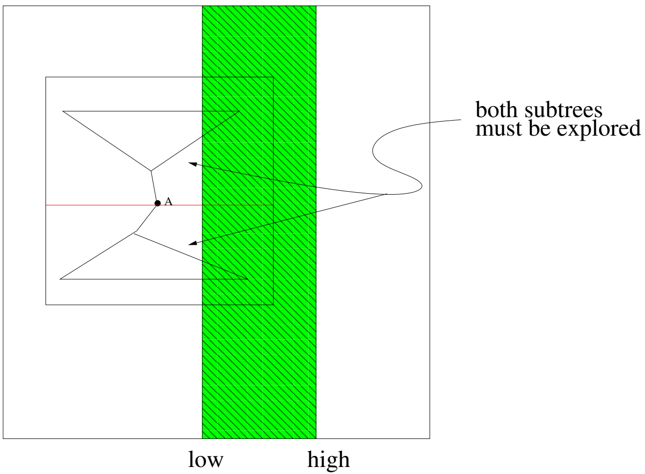
The algorithm: main loop



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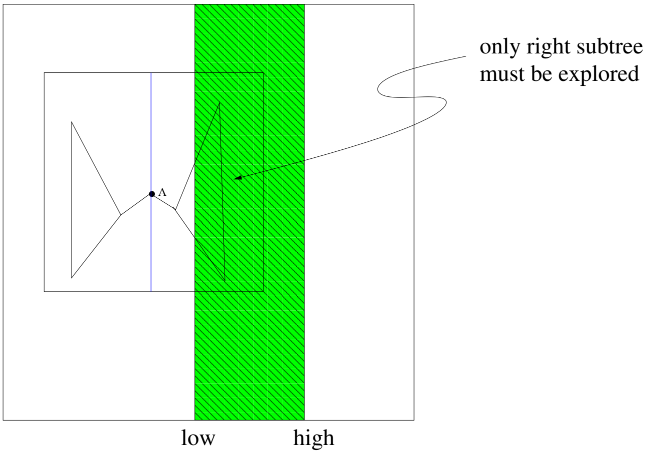

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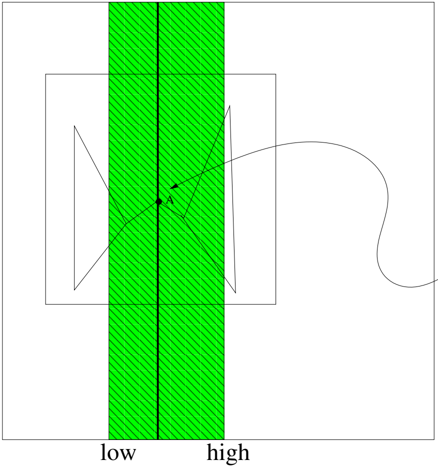
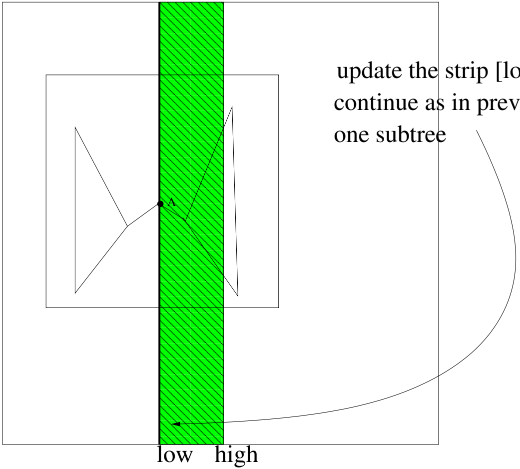


find the rank of the element along coordinate j = count how many elements are below

The algorithm: main loop



The algorithm

procedure KD-SELECT(T, i, j)
    Q.PUSH(T)
    $low \leftarrow 0$; $high \leftarrow 1$
    $found \leftarrow$ **false**
    while $\neg Q$.EMPTY() $\wedge \neg found$ **do**
        $t \leftarrow Q$.POP()
        if $t.discr \neq j$ **then**
            Q.PUSH($t.left$); Q.PUSH($t.right$)
        else
            ... next slide ...
    ▷ $found =$ **true** or the “strip” $[low, high]$ contains
    ▷ the i-th record along coordinate j
    ...

The algorithm

while $\neg$Q.EMPTY() $\wedge \neg \mathrm{found}$ **do**
  $\mathrm{t} \leftarrow$ Q.POP()
  if $\mathrm{t.discr} \neq \mathrm{j}$ **then** ...
  else $\triangleright$ $\mathrm{t.discr} = \mathrm{j}$
    $z \leftarrow \mathrm{t.key}[\mathrm{j}]$
    if $z \in [\mathrm{low}, \mathrm{high}]$ **then**
      $\triangleright$ BELOW returns the number of points x in T such that $x_j \leqslant z$
      $\mathrm{r} \leftarrow$ BELOW$(\mathrm{T}, \mathrm{j}, z)$
      if $\mathrm{i} < \mathrm{r}$ **then** $\mathrm{high} \leftarrow z$
      else if $\mathrm{i} > \mathrm{r}$ **then** $\mathrm{low} \leftarrow z$
      else $\mathrm{found} \leftarrow$ **true**
    if $z \leqslant \mathrm{low}$ **then** Q.PUSH$(\mathrm{t.right})$
    if $\mathrm{high} \leqslant z$ **then** Q.PUSH$(\mathrm{t.left})$

Analysis

Hypothesis for the analysis:

- The n records are independently drawn from a continuous distribution in $[0, 1]^K$ (standard probability model for random K-d tree)
- The sought rank i is random, with uniform probability in $[1..n]$
- The given coordinate j is also random, with uniform probability in $[1..K]$

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Analysis

Five key observations

1. The number of visited nodes in the main loop is at most the number of nodes visited by an orthogonal range search with the strip $[low, high]$
2. The cost of a call to BELOW is that of a partial match with a single specified coordinate
3. The expected number of calls to BELOW is $\Theta(\log n)$
4. The main loop finds the sought point when the node discriminates along j-th coordinate or the strip $[low, high]$ contains it and no point that discriminates with respect to j
5. The strip contains $\Theta(1)$ points on average

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Analysis

To achieve a good expected performance for a call to BELOW, it is necessary that each node contains the size of the subtree rooted at that tree

procedure BELOW(T, j, z)
    if $\mathsf{T} = \square$ **then return** 0
    if $\mathsf{T}.discr \neq j$ **then**
        $c \leftarrow [\![\mathsf{T}.key[j] \leqslant z]\!]$
        return $\text{BELOW}(z, j, \mathsf{T}.left) + \text{BELOW}(z, j, \mathsf{T}.right) + c$
    else
        if $z < \mathsf{T}.key[j]$ **then return** $\text{BELOW}(z, j, \mathsf{T}.left)$
        else return $\mathsf{T}.left.size + \text{BELOW}(z, j, \mathsf{T}.right)$

Analysis

- The expected cost of the second and third phases (if needed) is $\Theta(1)$ (Observation #5)
- The expected cost of the main loop, without counting the cost of calls to BELOW is $\Theta(n^\alpha)$ (Observation #1), where $\alpha = \alpha(K)$ depends on the type of K-d tree; for any K and any variant of K-d trees

  $$1 - \frac{1}{K} \leqslant \alpha(K) < 1$$

  For instance $\alpha(2) \approx 0.56$ for standard K-d trees
- The expected cost of a call to BELOW is $\Theta(n^\alpha)$ (Observation #2)
- The expected cost of the algorithm is $\Theta(n^\alpha \log n)$ (Observations #1 – #3)

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Final remarks

- A simple algorithm with sublinear expected cost
- It can easily be extended to many other multidimensional data structures
- Very little overhead: storing the size of each subtree is not very space consuming and it can also be sucessfully used for balancing (e.g., randomized relaxed K-d trees)
- Experiments show that it is competitive in practice compared to alternative solutions, for reasonably low dimensions (when K grows, $\alpha(K) \to 1$)

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Merci beaucoup!