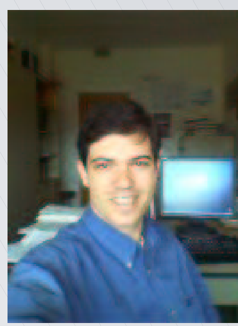
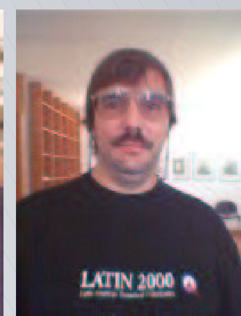
Adaptive Sampling for Quickselect





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Introduction

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- It partitions the given array around a pivot and continues into the appropriate subarray
- Quickselect is efficient: e.g. (Knuth, 1971)

$$C_{n,m} = m_0(\alpha) \cdot n + o(n) = 2(1 + \mathcal{H}(\alpha)) \cdot n + o(n)$$
$$= (2 - 2(\alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha))) \cdot n + o(n),$$

with $0 \leq \alpha = \frac{m}{n} \leq 1$.

The Algorithm

```
Elem quickselect(vector<Elem>& A,
                  int m) {
    int l = 0; int u = A.size() - 1;
    int k, p;
    while (l  ≤ u) {
       p = get_pivot(A, l, u, m);
       swap(A[p], A[l]);
       partition(A, l, u, k);
       if (m < k) u = k-1;
       else if (m > k) l = k+1;
       else return A[k];
}   }
```

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$$m_1(\alpha) = 2 + 3\alpha(1 - \alpha).$$

- For all α, $0 \leq \alpha \leq 1$, $m_0(\alpha) \leq m_1(\alpha)$. Also, $\overline{m}_0 = 3$ and $\overline{m}_1 = 2.5$.

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- In general: $r(\alpha)$ = rank of the pivot within the sample, when selecting the m-th out of n elements and $\alpha = m/n$
- Divide $[0, 1]$ into ℓ intervals with endpoints

$$0 = a_0 < a_1 < a_2 < \cdots < a_\ell = 1$$

and let r_k denote the value of $r(\alpha)$ for α in the k-th interval

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- "Proportion-from"-like strategies: $\ell = s$ and $r_k = k$, but the endpoints of the intervals $a_k \neq k/s$
- A sampling strategy is symmetric if

$$r(\alpha) = s + 1 - r(1-\alpha).$$

The Recurrence

- Probability that the r-th element in a sample of size s is the j-th element of the n given elements:

$$\pi_{n,j}^{(s,r)} = \frac{\binom{j-1}{r-1}\binom{n-j}{s-r}}{\binom{n}{s}},$$

$$1 \leq r \leq s \leq n, \quad 1 \leq j \leq n.$$

The Recurrence

- Average number of comparisons $C_{n,m}$ to select the m-th out of n:

$$C_{n,m} = n + \Theta(1) + \sum_{j=m+1}^{n} \pi_{n,j}^{(s,r)} \cdot C_{j-1,m} + \sum_{j=1}^{m-1} \pi_{n,j}^{(s,r)} \cdot C_{n-j,m-j}.$$

A General Theorem

Theorem 1. *Let* $f(\alpha) = \lim_{n\to\infty, m/n\to\alpha} \frac{C_{n,m}}{n}$. *Then*

$$f(\alpha) = 1 + \frac{s!}{(r(\alpha)-1)!(s-r(\alpha))!} \times \left[\int_\alpha^1 f\left(\frac{\alpha}{x}\right) x^{r(\alpha)}(1-x)^{s-r(\alpha)}\,dx + \int_0^\alpha f\left(\frac{\alpha-x}{1-x}\right) x^{r(\alpha)-1}(1-x)^{s+1-r(\alpha)}\,dx \right].$$

Two Elementary Facts

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- If $r(\alpha)$ is symmetric then $f(\alpha) = f(1-\alpha)$.
- Let $r_0 = \lim_{\alpha\to 0} r(\alpha)$. Then

$$\lim_{\alpha\to 0} f(\alpha) = \frac{s+1}{s+1-r_0}.$$

In proportion-from strategies $r_0 = 1$; hence, $f(0) = 1 + 1/s$, while for median-of-$(2t+1)$, we have $m_t(0) = 2$

The General Differential Equation

Denote f_k the restriction of $f(\alpha)$ to the k-th interval of $[0,1]$.

Lemma 1. *For any adaptive sampling strategy*

$$\frac{d^{s+2}}{d\alpha^{s+2}} f_k(\alpha) = \frac{(-1)^{s+1-r_k} \cdot s!}{\alpha^{s+1-r_k}(r_k-1)!} \frac{d^{r_k+1}}{d\alpha^{r_k+1}} f_k(\alpha) + \frac{s!}{(1-\alpha)^{r_k}(s-r_k)!} \frac{d^{s+2-r_k}}{d\alpha^{s+2-r_k}} f_k(\alpha).$$

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- We do not know the initial values of the f_k's and their derivatives
- Plug the general form of the f_k's back into the integral equation(s) and solve for the unknown constants

Proportion-from-2

- The differential equation is

$$\frac{d^2\phi_1}{dx^2} - \frac{2}{1-x}\frac{d\phi_1}{dx} - \frac{2}{x^2}\phi_1 = 0$$

with $\phi_1(x) = f_1''(x)$ and $f_2(x) = f_1(1-x)$.

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- The solution is

$$f_1(x) = a\left((x-1)\ln(1-x) + \frac{x^3}{6} + \frac{x^2}{2} - x\right) - b(1 + \mathcal{H}(x)) + cx + d.$$

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- The grand-average: $C_n = \overline{f} \cdot n + o(n)$, with

$$\overline{f} = 2.598\ldots$$

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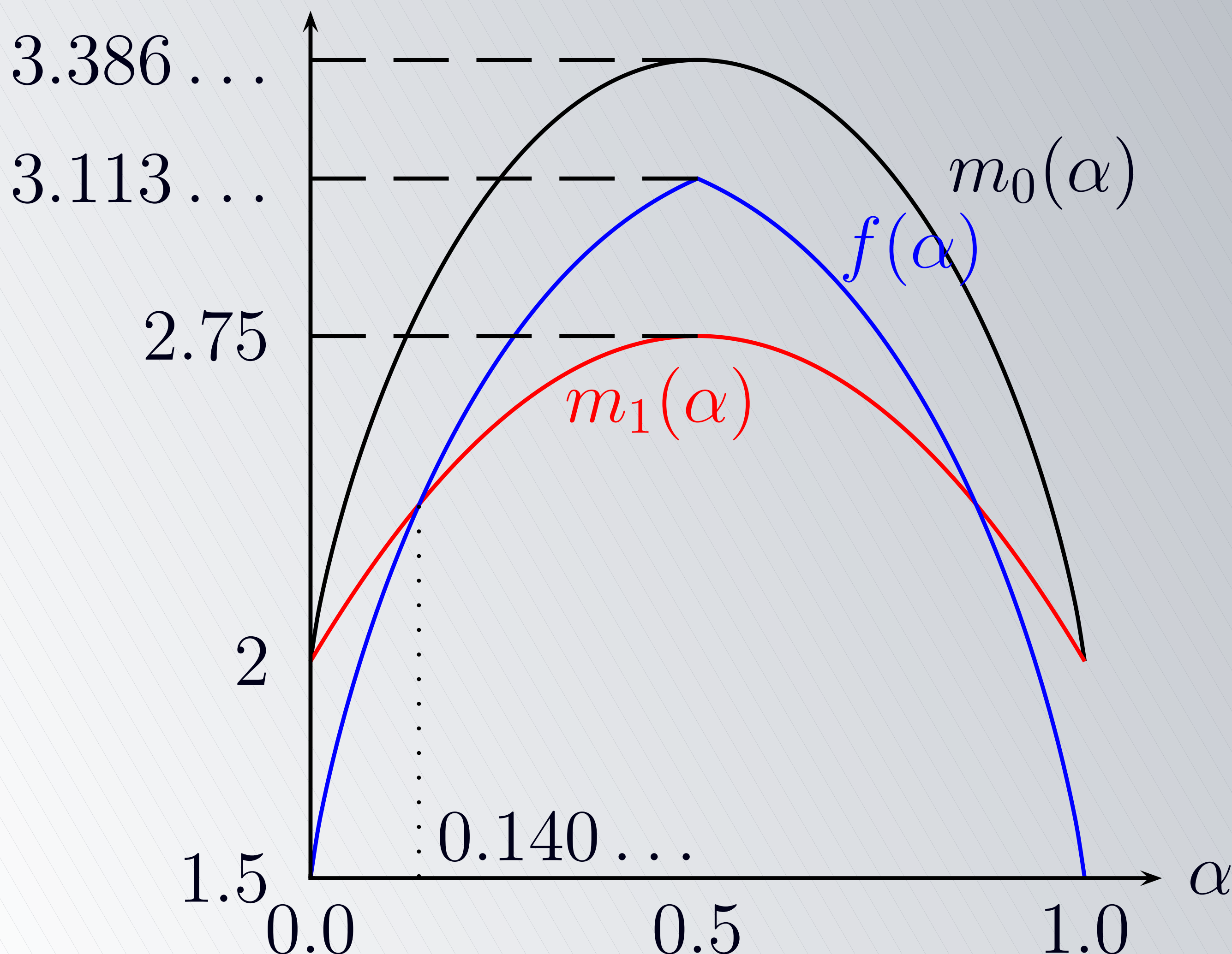

Proportion-from-3

For proportion-from-3,

$$f_1(x) = -C_0(1+\mathcal{H}(x)) + C_1 + C_2 x + C_3 K_1(x) + C_4 K_2(x),$$
$$f_2(x) = -C_5(1+\mathcal{H}(x)) + C_6 x(1-x) + C_7,$$

with

$$K_1(x) = \cos(\sqrt{2}\ln x)\cdot\sum_{n\geq 0} A_n x^{n+4} + \sin(\sqrt{2}\ln x)\cdot\sum_{n\geq 0} B_n x^{n+4},$$
$$K_2(x) = \sin(\sqrt{2}\ln x)\cdot\sum_{n\geq 0} A_n x^{n+4} - \cos(\sqrt{2}\ln x)\cdot\sum_{n\geq 0} B_n x^{n+4}.$$

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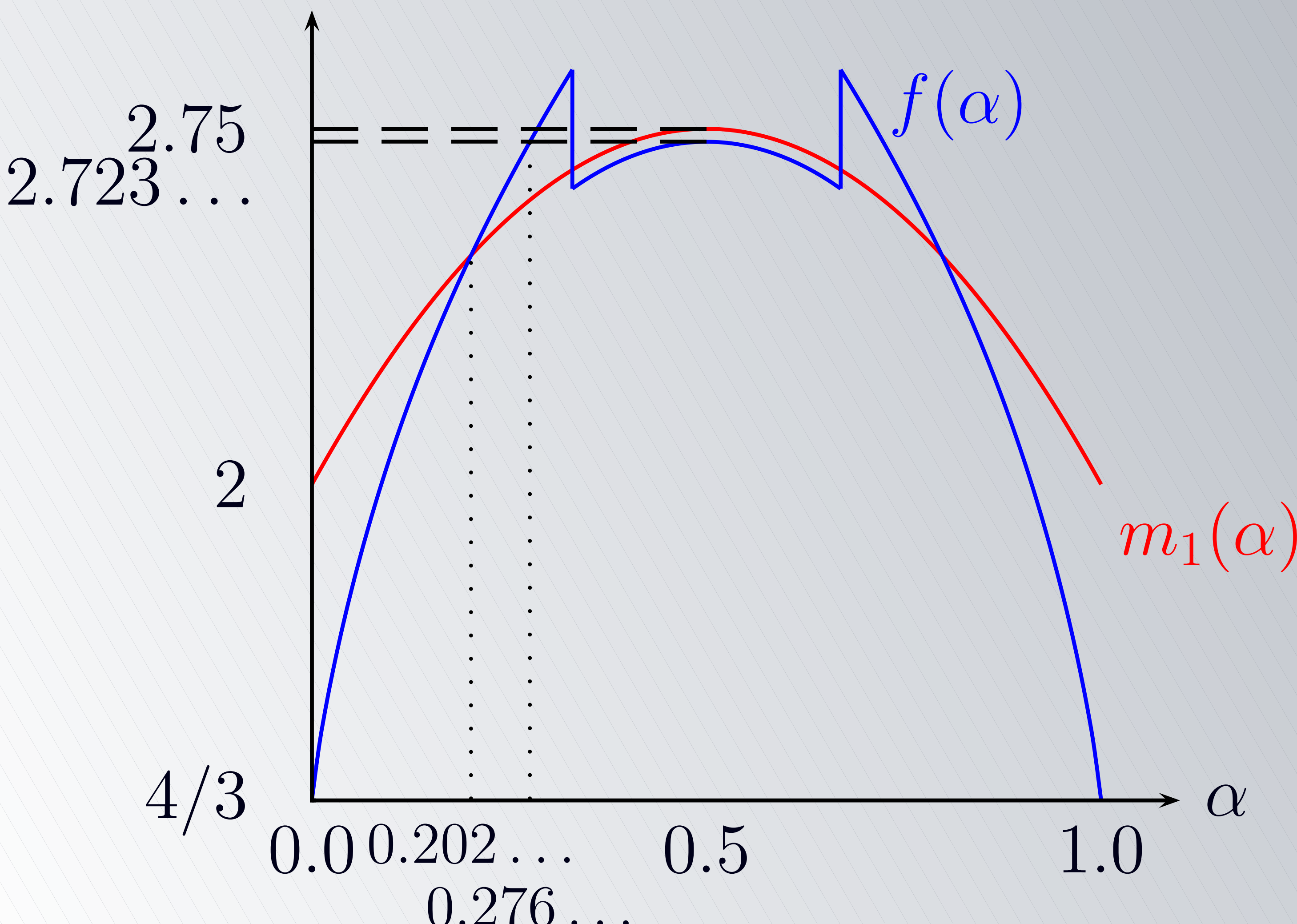
$$\overline{f} = 2.421\ldots$$

Proportion-from-3: Batfind



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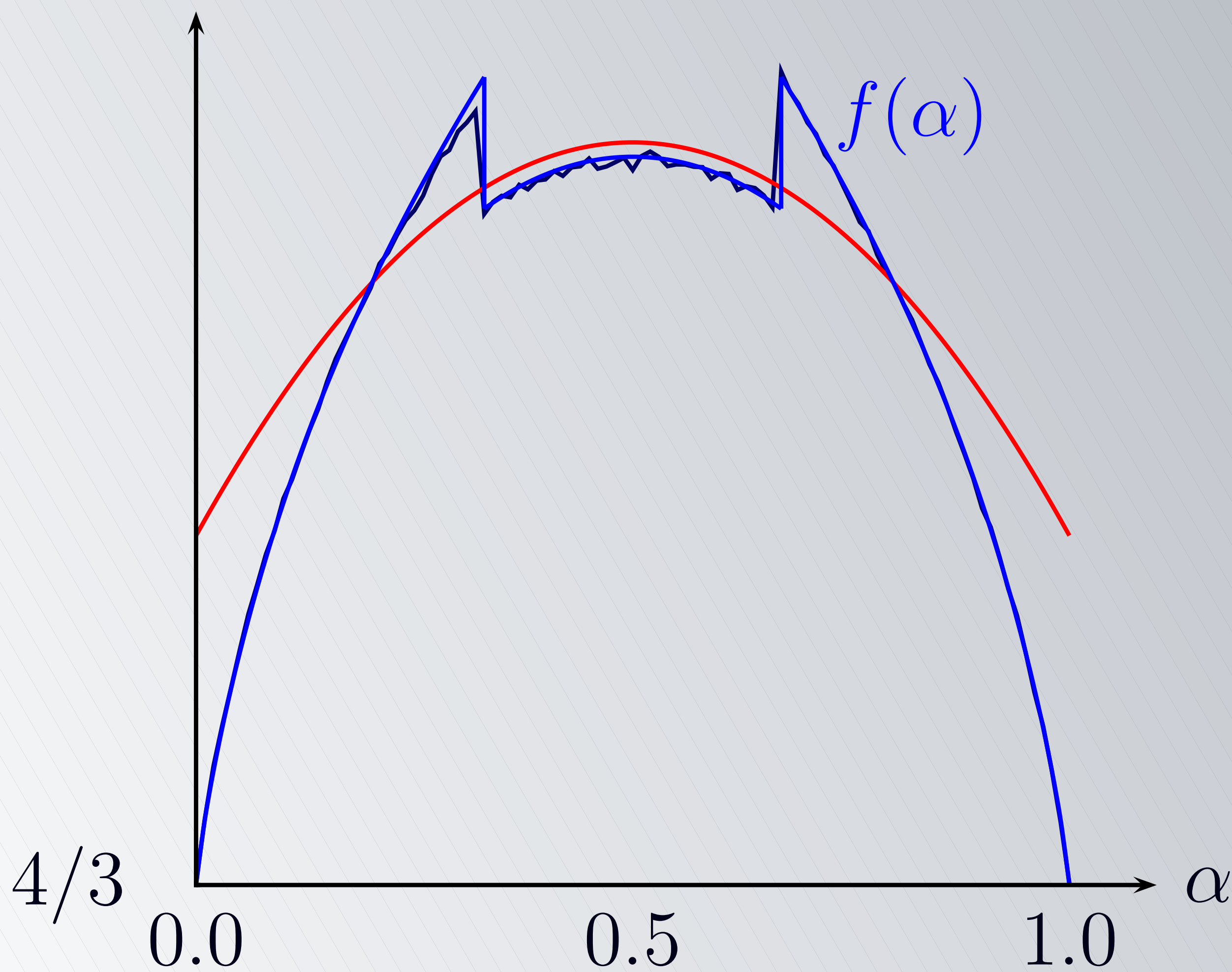

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- Same differential equation, same f_i's, with $C_i = C_i(\nu)$
- If $\nu \to 0$ then $f_\nu \to m_1$ (median-of-three)
- However, if $\nu \to 1/2$ then f_ν behaves like proportion-from-2, but it is not the same

The optimal ν

Theorem 2. *There exists a value ν^*, namely, $\nu^* = 0.182\ldots$, such that for any ν, $0 < \nu < 1/2$, and any α,*

$$f_{\nu^*}(\alpha) \leq f_\nu(\alpha).$$

Furthermore, ν^* is the unique value of ν such that f_ν is continuous,i.e.,

$$f_{\nu^*,1}(\nu^*) = f_{\nu^*,2}(\nu^*).$$

More on ν-find

- If $\nu > \tilde{\nu} = 0.268\ldots$ then f_ν has two absolute maxima at $\alpha = \nu$ and $\alpha = 1 - \nu$; otherwise there is one absolute maximum at $\alpha = 1/2$

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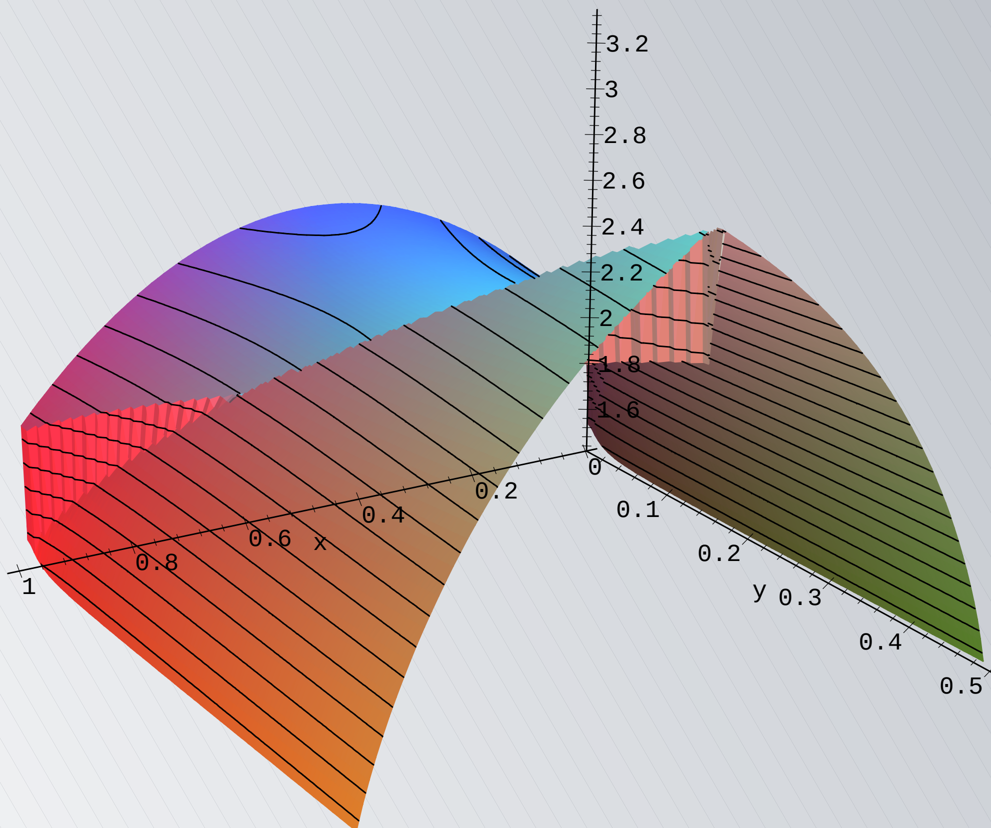
- If $\nu > \tilde{\nu} = 0.268\ldots$ then f_ν has two absolute maxima at $\alpha = \nu$ and $\alpha = 1 - \nu$; otherwise there is one absolute maximum at $\alpha = 1/2$
- Obviously, the value ν^* minimizes the maximum

$$f_{\nu^*}(1/2) = 2.659\ldots$$

and the mean

$$\overline{f}_{\nu^*} = 2.342\ldots.$$

More on ν-find



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- If $\nu \leq \nu'_m = 0.364\ldots$ then ν-find beats median-of-3 to find the median: $f_\nu(1/2) \leq 11/4$
- If $\nu \leq \nu' = 0.219\ldots$ then ν-find beats median-of-3 for all ranks: $f_\nu(\alpha) \leq m_1(\alpha)$

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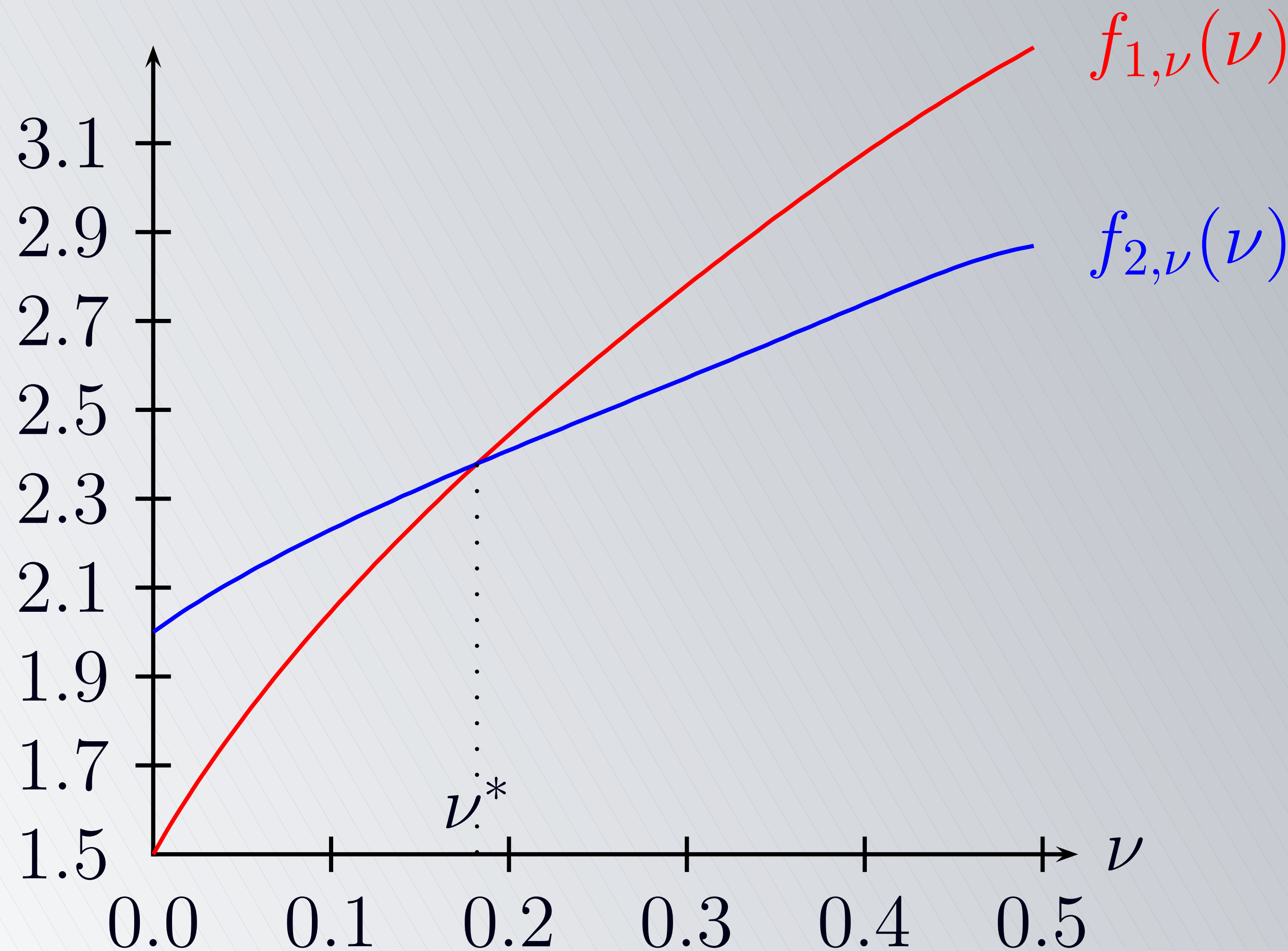

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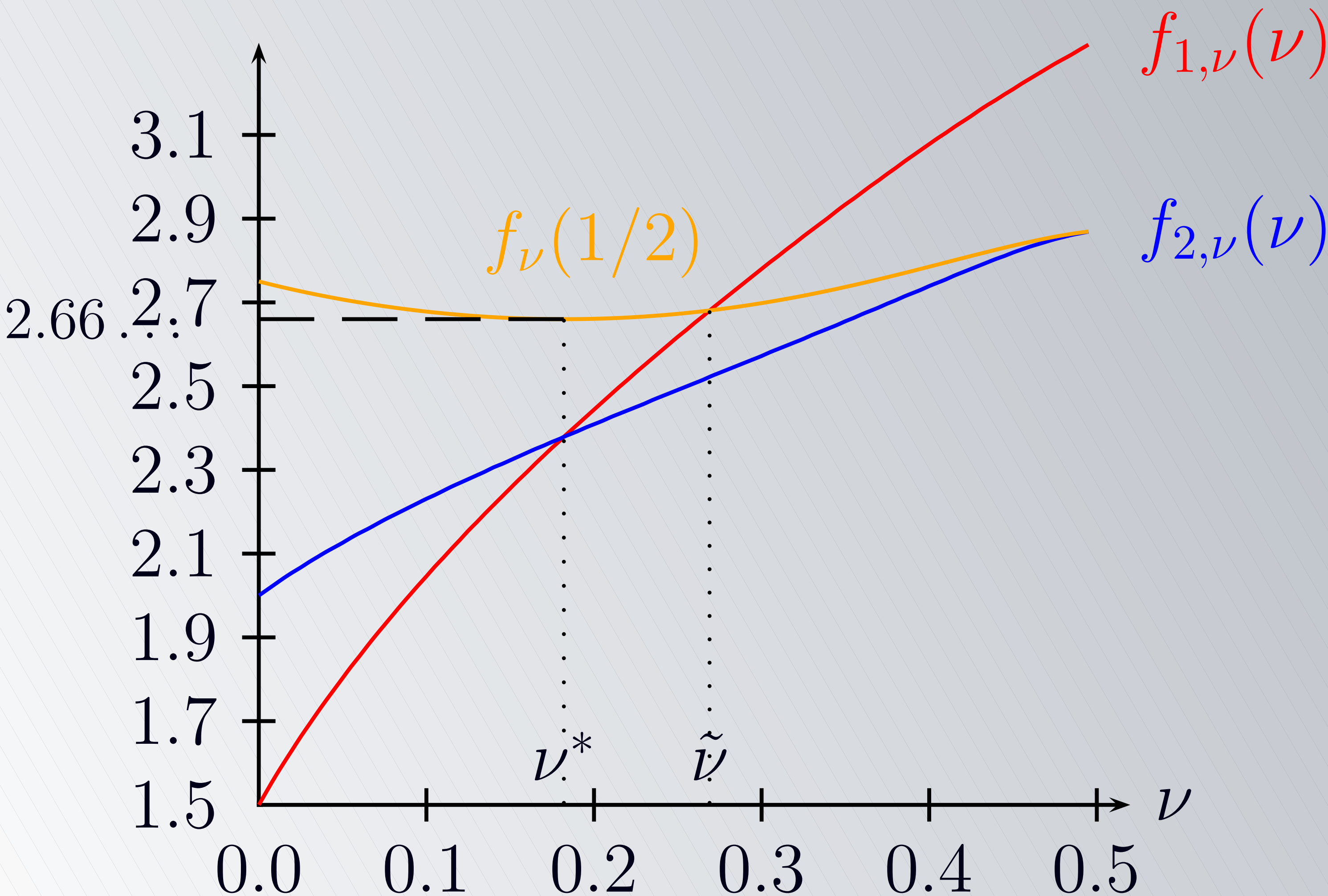

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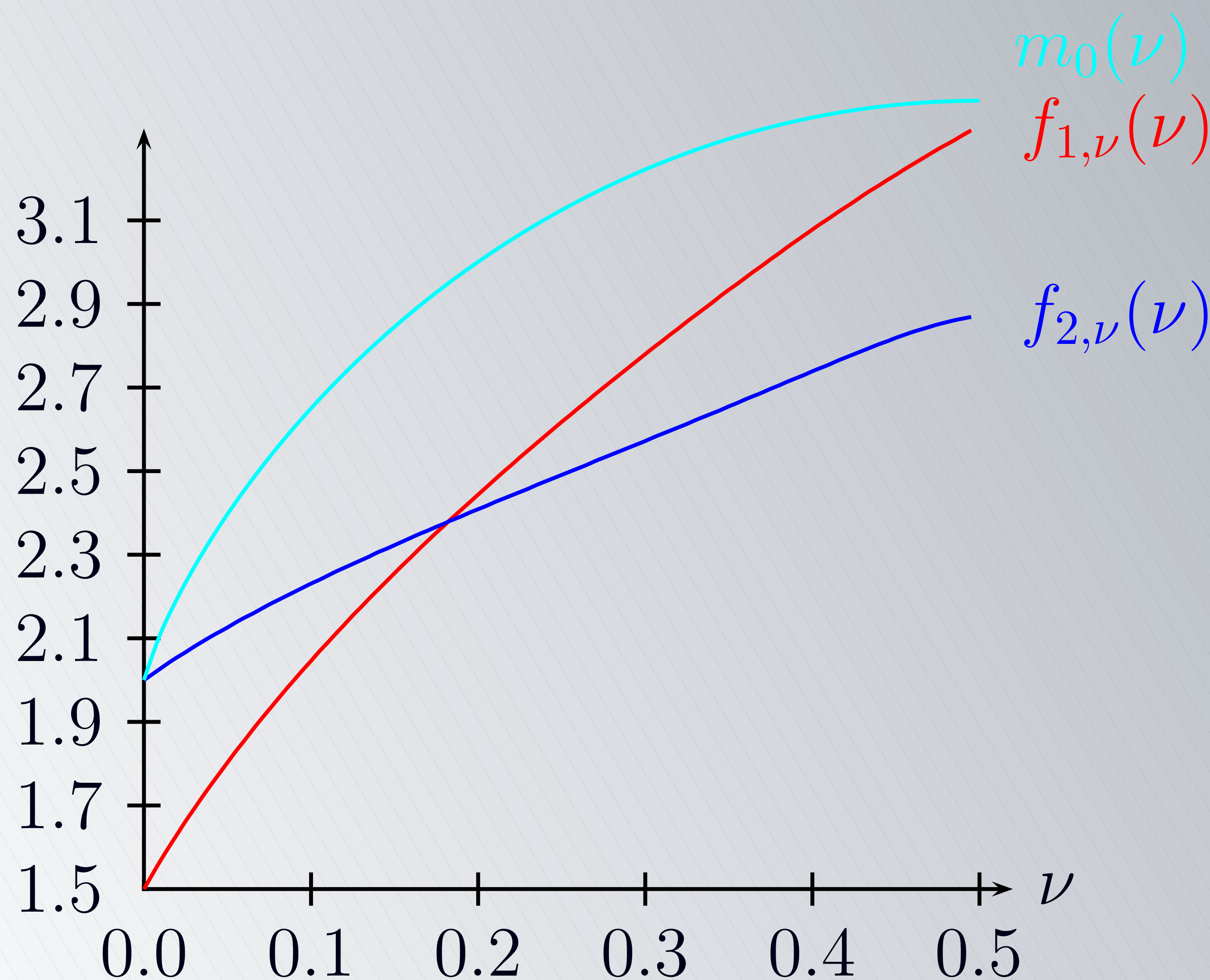

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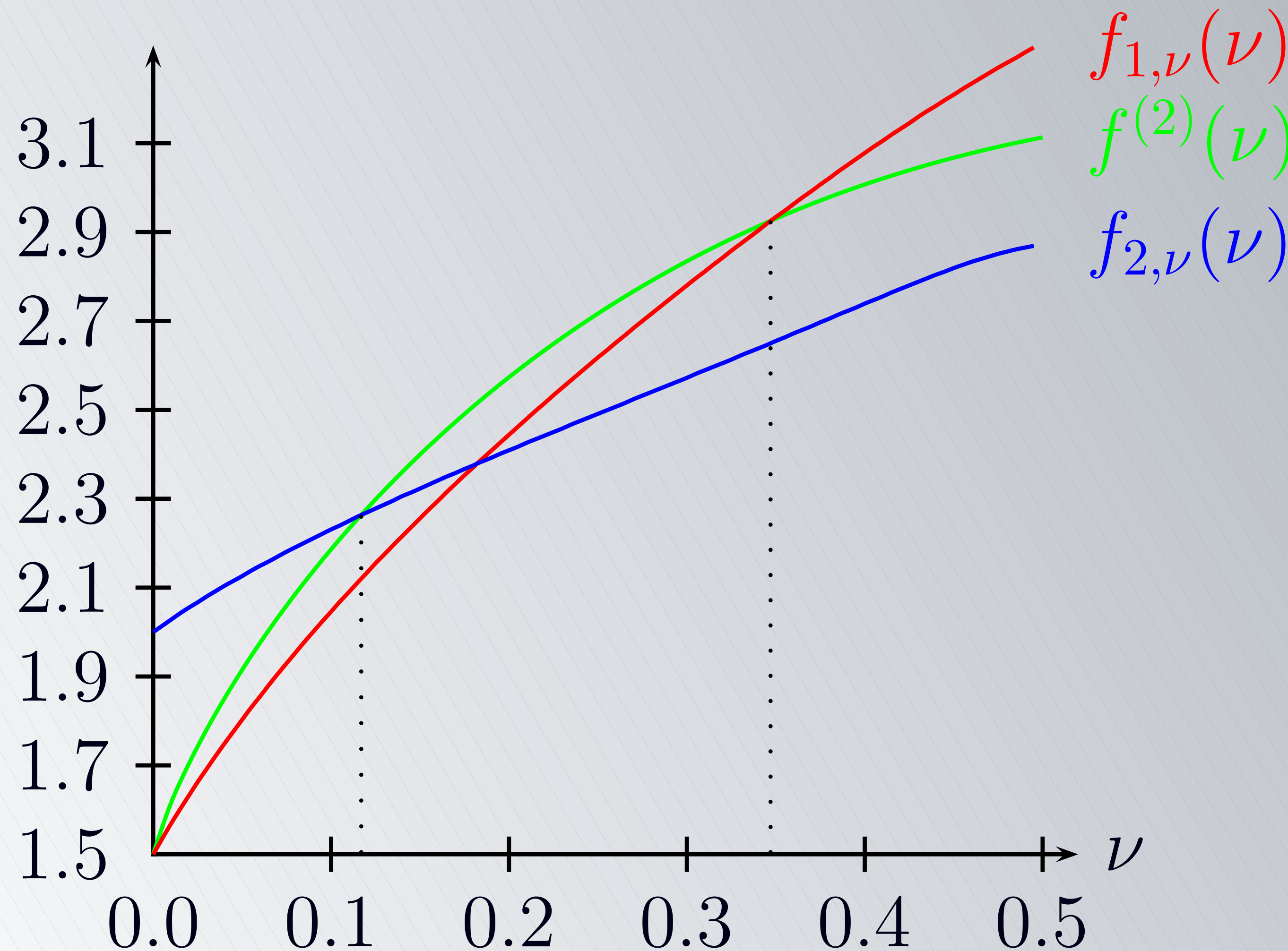

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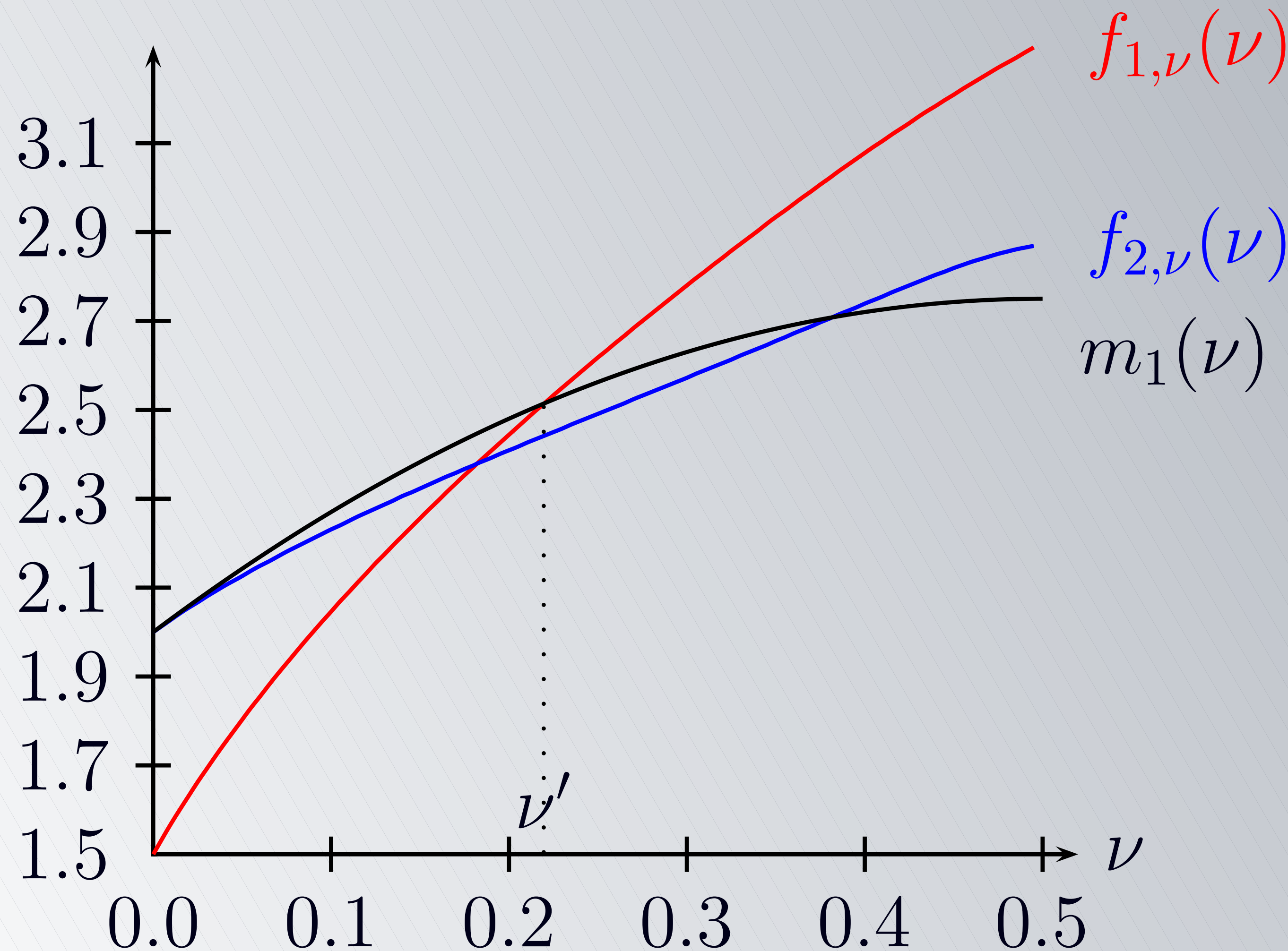

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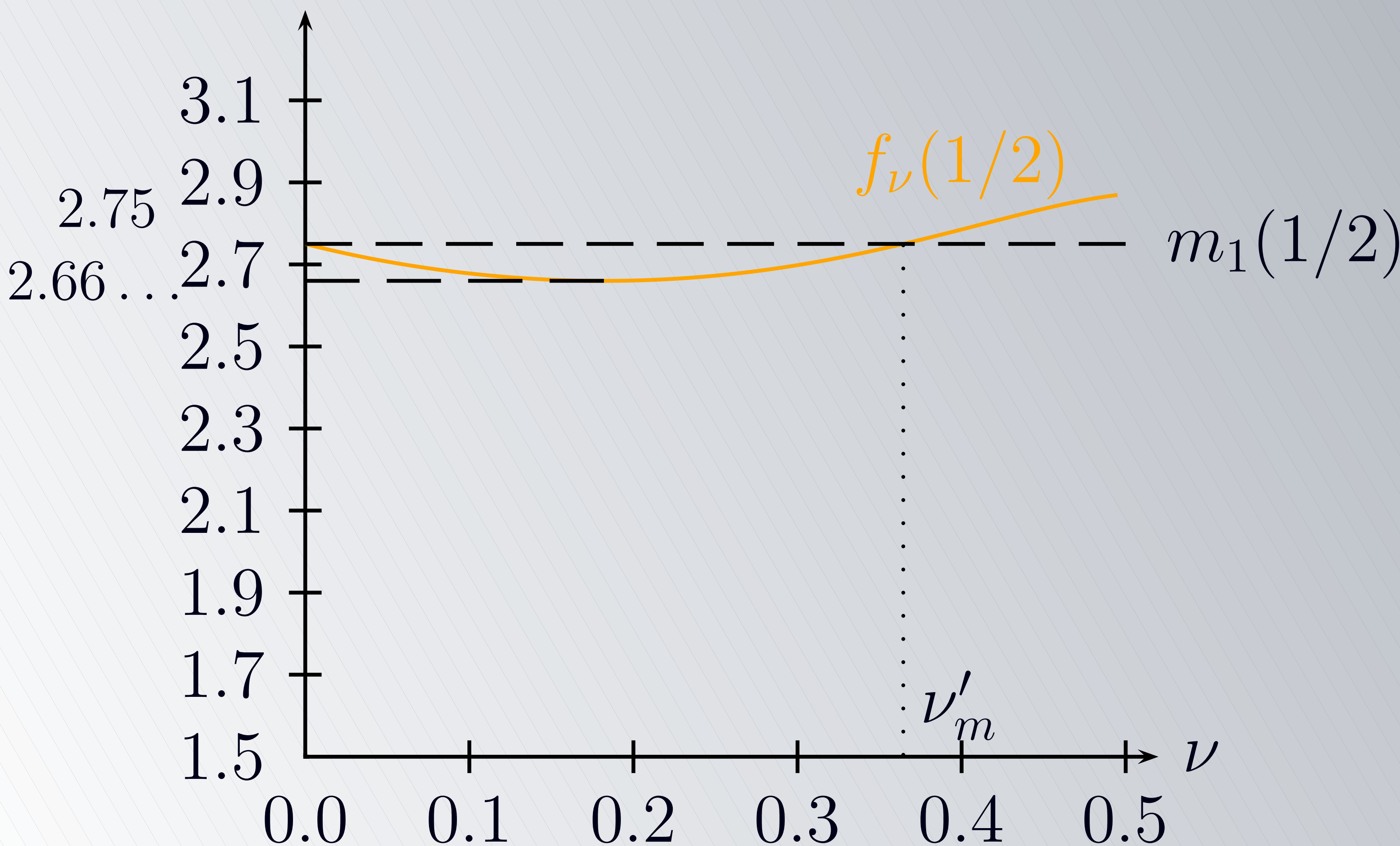


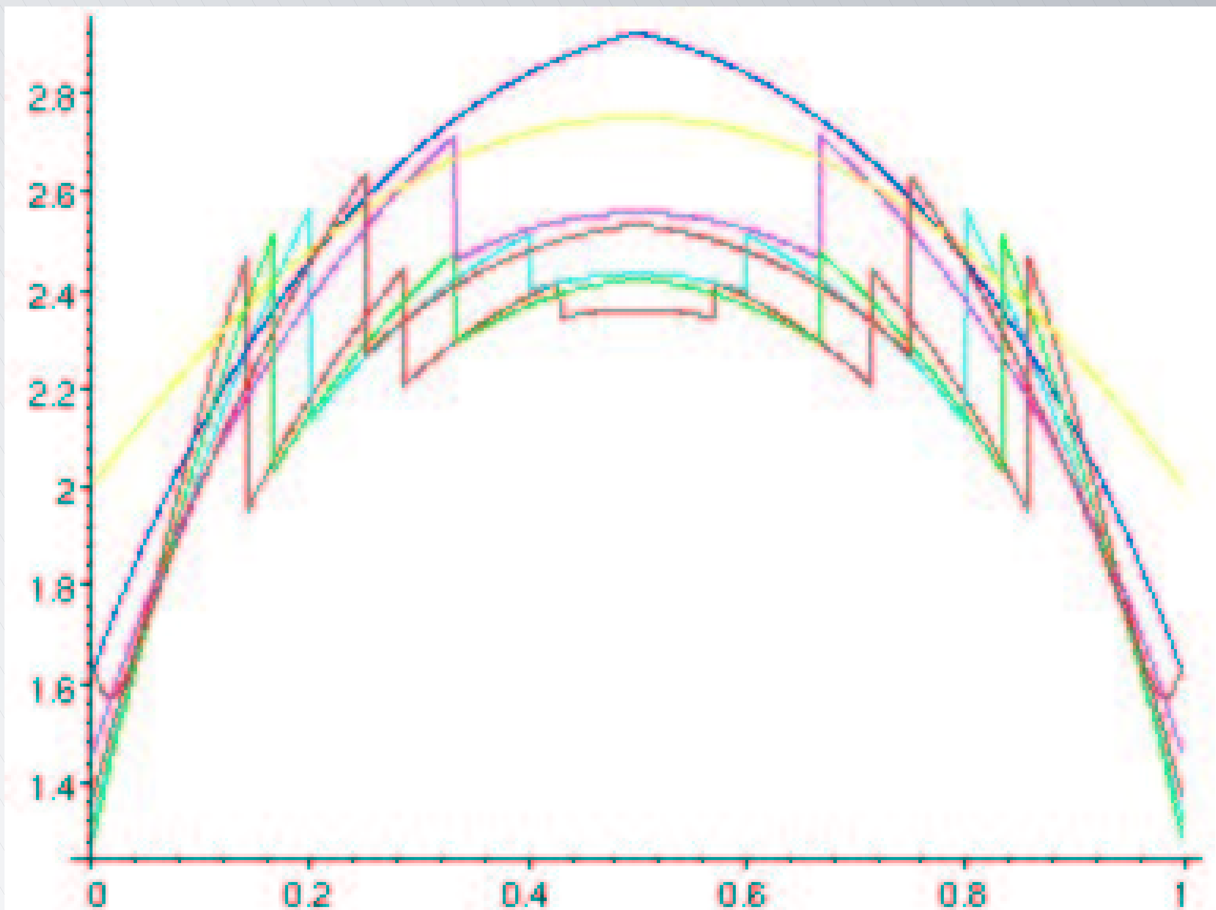
More on ν-find

- We have investigated the average total cost of ν-find

$$\lambda_1 \cdot \text{\# of comparisons} + \lambda_2 \cdot \text{\# of exchanges}$$

- The values of ν^* (optimum), ν' (ν-find beats median-of-three), etc. now depend on λ_2/λ_1; for instance, if $\lambda_2/\lambda_1 = \infty$ we minimize the average number of exchanges with $\nu^* = 0.43\ldots$

Proportion-from-s: Sharkfind



Proportion-from-s: $s \to \infty$

Theorem 3. *Let* $f_s(\alpha) = \lim_{n\to\infty, m/n\to\alpha} \frac{C_{n,m}}{n}$ *when using samples of size* s. *Then for any adaptive sampling strategy such that* $\lim_{s\to\infty} r(\alpha)/s = \alpha$

$$f_\infty(\alpha) = \lim_{s\to\infty} f_s(\alpha) = 1 + \min(\alpha, 1-\alpha).$$

This is theoretically optimal for comparison-based selection algorithms.