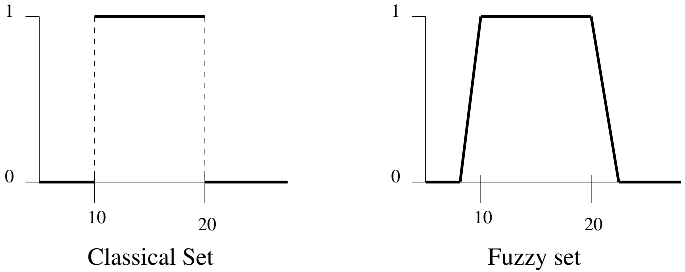
Possibilistic model: goals

- Used to represent language imprecision
- Experts formalize their knowledge as qualitative descriptions (words instead of exact values)
- For example: "**temperature** is **high**"
- Two components:
  - Variables defined in a domain (Universe of discourse, $\mathcal{U}$)
  - Sets of qualitative terms about the variables (linguistic labels)

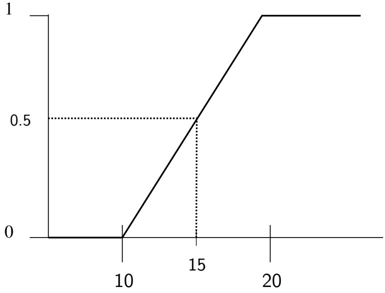
Fuzzy sets

- A linguistic label does not represent a set in a classical sense
- Depending on the specific value, the belief on the membership to the set changes (It is difficult to establish the frontier point)
- **Fuzzy sets** represent the membership as a continuous value
- This idea can be used to model language imprecision



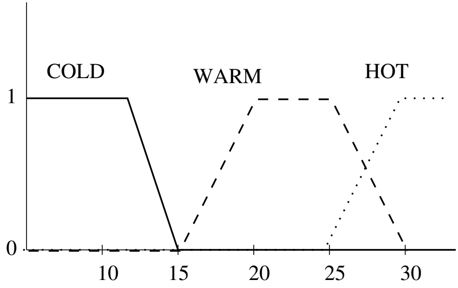
Fuzzy sets/Fuzzy logic

- Possibilistic logic is based on fuzzy sets (fuzzy logic)
- A linguistic label is identified with a fuzzy set
- Fuzzy sets are defined by their possibility function ($\pi_A : \mathcal{U} \longrightarrow [0,1]$)
- Each fuzzy set maps any value in a domain to a degree of membership using the possibility function



Fuzzy facts

- The knowledge about the domain will be represented using *fuzzy facts*
- Being X the variable over a domain $\mathcal{U}$ and A a fuzzy set defined over $\mathcal{U}$, we can say [X is A]
- For example: [Temperature is warm], [Temperature is cold]



Fuzzy Connectives

- In order to combine fuzzy facts and to create more complex expressions fuzzy logic connectives are used
- They are a *continuous extension* of the classical logic connectives
- They combine the possibility values of the facts to compute the *possibility value* of the combination
- Three connectives are used: conjunction, disjunction and negation

Fuzzy conjunction: T-norm

The function that computes the conjunction is the T-norm, it is defined as $T(x, y) : [0, 1] \times [0, 1] \to [0, 1]$ and has the following properties:

1. Commutative, $T(a, b) = T(b, a)$
2. Associative, $T(a, T(b, c)) = T(T(a, b), c)$
3. $T(0, a) = 0$
4. $T(1, a) = a$
5. It is an increasing function, $T(a, b) \leq T(c, d)$ if $a \leq c$ and $b \leq d$

For example

- $T(x, y) = \min(x, y)$
- $T(x, y) = x \cdot y$
- $T(x, y) = \max(0, x + y - 1)$

Fuzzy disjunction

The function that computes the conjunction is the T-conorm, it is defined as $S(x, y) : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and has the following properties:

1. Commutative, $S(a, b) = S(b, a)$
2. Associative, $S(a, S(b, c)) = S(S(a, b), c)$
3. $S(0, a) = a$
4. $S(1, a) = 1$
5. It is an increasing function, $S(a, b) \leq S(c, d)$ if $a \leq c$ and $b \leq d$

For example:

- $S(x, y) = \max(x, y)$
- $S(x, y) = x + y - x \cdot y$
- $S(x, y) = \min(x + y, 1)$

Fuzzy negation

The function that negates a possibility value is the strong negation and it is defined as $N(x) : [0,1] \rightarrow [0,1]$ and has the following properties:

1. $N((N(a)) = a$
2. It is a decreasing function, $N(a) \geq N(b)$ if $a \leq b$

For example:

- $N(x) = 1 - x$
- $N(x) = \sqrt{1 - x^2}$
- $N_t(x) = \frac{1-x}{1+t \cdot x}$ $t > 1$

Fuzzy combining

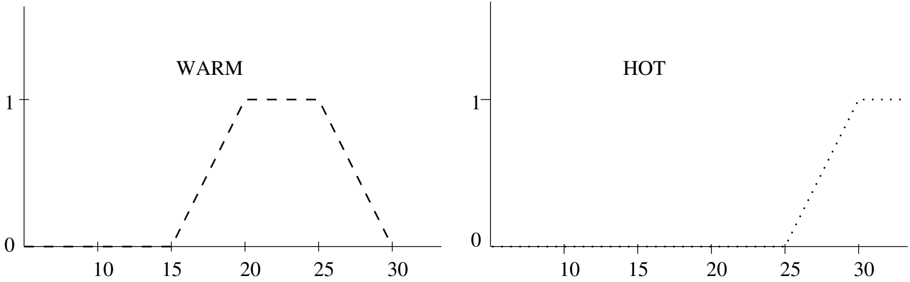
- The result of combining fuzzy facts is a new fuzzy set
- The characteristics of this set depend on if the combination is of facts from the same universe of the discourse or not
- If the facts are from the same universe the new fuzzy set has a characteristic function of one dimension

$$\pi_{A\otimes B}(\mathcal{U}) = f(\pi_A(\mathcal{U}), \pi_B(\mathcal{U}))$$

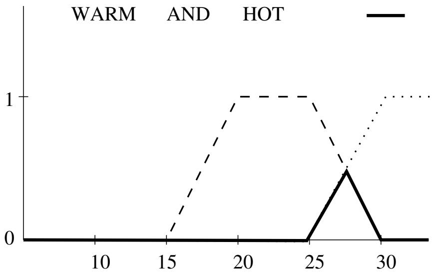
- If the facts are from different universes the characteristic function has as dimensions as different universes are the facts from

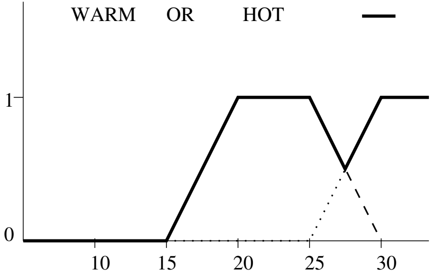
$$\pi_{A\otimes B}(\mathcal{U}, \mathcal{V}) = f(\pi_A(\mathcal{U}), \pi_B(\mathcal{V}))$$

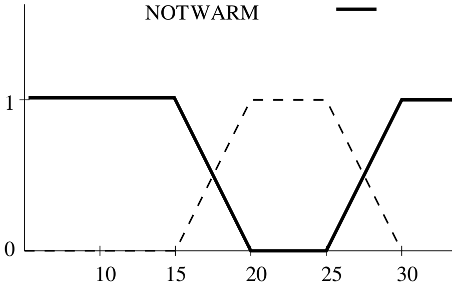
Fuzzy sets from the same universe



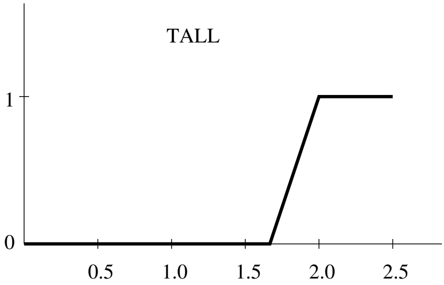
Fuzzy sets from the same universe

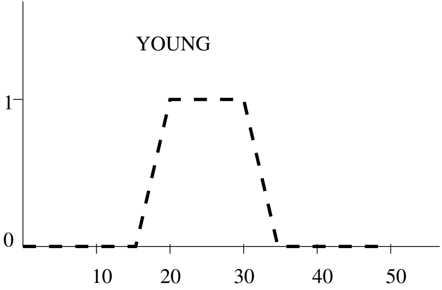




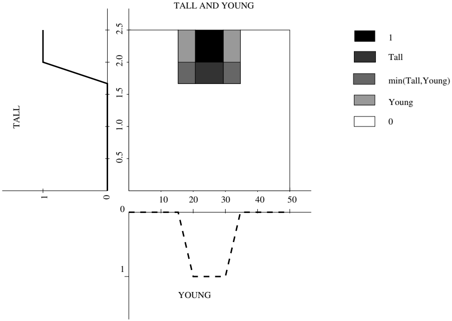


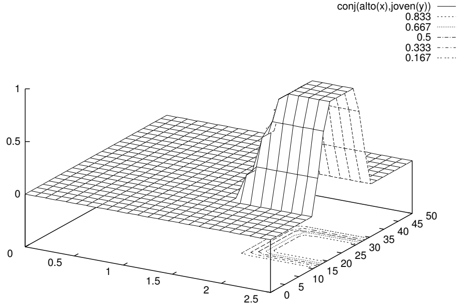
Fuzzy sets from different universes



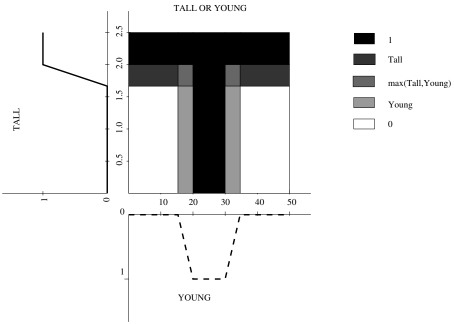


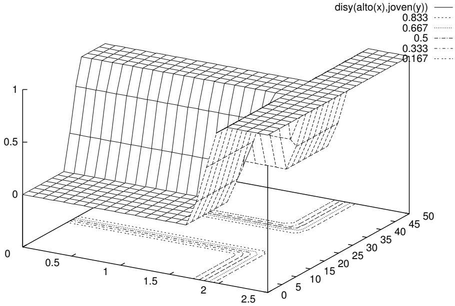
Fuzzy sets from different universes





Fuzzy sets from different universes





Inference in fuzzy logic

- Implication is included as combination operator
- A semantic for the implication is defined
- Two scenarios:
    - Inference with fuzzy data (definition of a combination function for the implication, *modus ponens* rule)
    - Inference with crisp data (definition of how to compute the possibility of antecedent and consequent)

Inference with crisp data

Two models are the most used

- **Mamdani inference model:** The consequent of a rule is a fuzzy fact, the inference consists in to apply the possibility value of the antecedent to the consequent
- **Sugeno inference model:** The consequent of a rule is a linear function of the possibility values of the antecedent, the value of this function is the result of the inference

Mamdani inference model

- **Antecedent evaluation:** Computation for each rule of the possibility value of the crisp values using the antecedent facts and their combination
- **Consequent evaluation:** Each consequent is weighted using the possibility value of its antecedent
- **Conclusion combination:** The conclusions are combined in a fuzzy set that represents the joint conclusion
- **Nitidification:** The crisp value corresponding to the conclusion is computed from the fuzzy (for example computing the center of mass of the fuzzy set)

$$CDG(f(x)) = \frac{\int_a^b f(x) \cdot x dx}{\int_a^b f(x) dx}$$

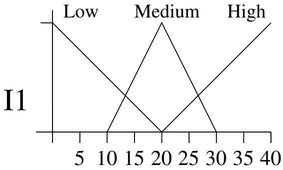
Fuzzy inference: Example (1)

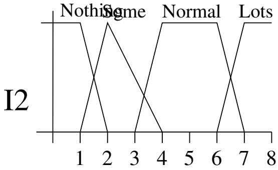
Assume that we have two input variables $I1$ and $I2$ and an output variable O that have the values defined by the following linguistic labels:

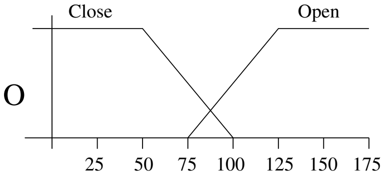
- $E1$=$\{$low,medium,high$\}$
- $E2$=$\{$nothing,some,normal,lots$\}$
- S=$\{$close,open$\}$

Fuzzy inference: Example (2)

These labels are represented by the following fuzzy sets







Fuzzy inference: Example (3)

Assume that we have the following rules:

R1. if ([I1=medium] or [I1=high]) y [I2=lots] then [O=close]

R2. if [I1=high] and [I2=normal] then [O=close]

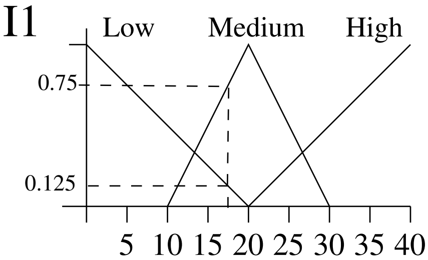
R3. if [I1=low] and not([I2=lots]) then [O=open]

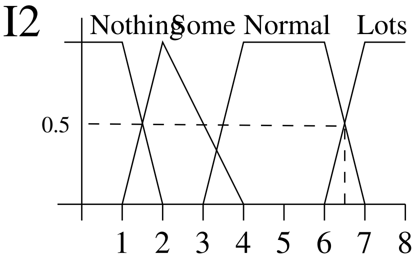
R4. if ([I1=low] or [I1=medium]) and [I2=some] then [O=open]

The combination functions are the minimum for the conjunction, the maximum for the disjunction and $1 - x$ for the negation and the crisp values for variables $I1$ and $I2$ are 17.5 and 6.5 respectively

Fuzzy inference: Example (4)

If we use the values on the fuzzy sets of variables $I1$ and $I2$ we obtain the following possibility values





Fuzzy inference: Example (5)

- If we evaluate the **rule R1** we have:

  [I1=medium] $= 0.75$, [I1=high] $= 0$, [I2=lots] $= 0.5 \Rightarrow \min(\max(0.75,0),0.5) = 0.5$
  So we have $0.5 \cdot$ [O=close]

- If we evaluate the **rule R2** we have:

  [I1=high] $= 0$, [I2=normal] $= 0.5 \Rightarrow \min(0,0.5) = 0$
  So we have $0 \cdot$ [O=close]

- If we evaluate the **rule R3** we have:

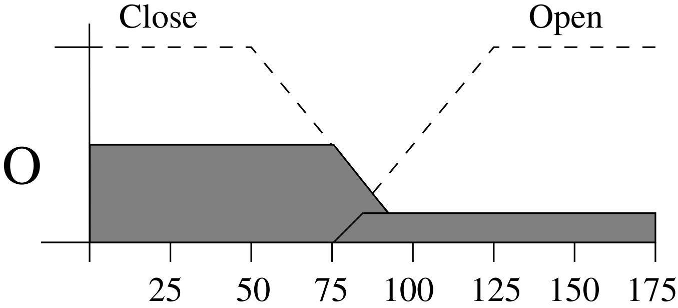
  [I1=low] $= 0.125$, [I1=lots] $= 0.5$, $\Rightarrow \min(0.125,1\text{-}0.5) = 0.125$
  So we have $0.125 \cdot$ [O=open]

- If we evaluate the **rule R4** we have:

  [I1=low] $= 0.125$, [I1=medium] $= 0.75$, [I2=some] $= 0 \Rightarrow \min(\max(0.125,0.75),0) = 0$
  So we have $0 \cdot$ [O=open]

Fuzzy inference: Example (6)

The fuzzy set resulting from the combination of the conclusion is:



Fuzzy inference: Example (7)

Now we compute the center of mass of the fuzzy set, this can be described with the function:

$$f(x) = \begin{cases} 0.5 & x \in [0 - 75] \\ (100 - x)/50 & x \in (75 - 93.75] \\ 0.125 & x \in (93.75 - 100] \end{cases}$$

Fuzzy inference: Example (8)

The CoM can be computed as:

$$\frac{\int_0^{75} 0.5 \cdot x \cdot dx + \int_{75}^{93.75} (100-x)/50 \cdot x \cdot dx + \int_{93.75}^{175} 0.125 \cdot x \cdot dx}{\int_0^{75} 0.5 \cdot dx + \int_{75}^{93.75} (100-x)/50 \cdot dx + \int_{93.75}^{175} 0.125 \cdot dx} =$$

$$\frac{0.25 \cdot x^2|_0^{75} + (x^2 - x^3/150)|_{75}^{93.75} + 0.0625 \cdot x^2|_{93.75}^{175}}{0.5 \cdot x|_0^{75} + (2 \cdot x - x^2/100)|_{75}^{93.75} + 0.125 \cdot x|_{93.75}^{175}} =$$

$$\frac{(0.25 \cdot 75^2 - 0) + ((93.75^2 - 93.75^3/150) - (75^2 - 75^3/150)) + (0.0625 \cdot 175^2 - 0.0625 \cdot 93.75^2)}{(0.5 \cdot 75 - 0) + ((2 \cdot 93.75 - 93.75^2/100) - (2 \cdot 75 - 75^2/100)) + (0.125 \cdot 175 - 0.125 \cdot 93.75)}$$

$$\frac{3254.39}{53.53} = 60.79$$