LIMITS OF LINEAR AND SEMIDEFINITE RELAXATIONS FOR COMBINATORIAL PROBLEMS

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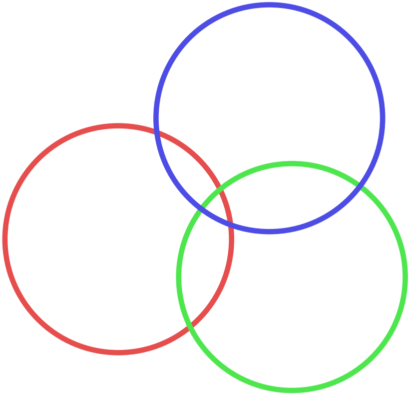
- linear and semidefinite programming
- approximation algorithms and computational complexity
- logic and finite model theory



Part I

LINEAR PROGRAMMING RELAXATIONS

Vertex cover

Problem:

Given an undirected graph $G = (V, E)$,
find the smallest number of vertices
that touches every edge.

Notation:

$$\mathrm{vc}(G).$$

Observe:

$A \subseteq V$ is a vertex cover of G
iff
$V \setminus A$ is an independent set of G

Linear programming relaxation

LP relaxation:

$$\begin{aligned}
&\text{minimize } \textstyle\sum_{u \in V} x_u \\
&\text{subject to} \\
&\quad x_u + x_v \geq 1 \quad \text{for every } (u, v) \in E, \\
&\quad x_u \geq 0 \qquad\quad\ \text{for every } u \in V.
\end{aligned}$$

Notation:

$$\mathrm{fvc}(G).$$

Approximation

Approximation:

$$\mathrm{fvc}(G) \leq \mathrm{vc}(G) \leq 2 \cdot \mathrm{fvc}(G)$$

Integrality gap:

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{fvc}(G)}$$

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Gap examples:

1. $\mathrm{vc}(K_{2n+1}) = 2n$,
2. $\mathrm{fvc}(K_{2n+1}) = \frac{1}{2}(2n+1)$.

LP tightenings

Add triangle inequalities:

minimize $\sum_{u \in V} x_u$

subject to

$$
\begin{array}{ll}
x_u + x_v \geq 1 & \text{for every } (u, v) \in E, \\
x_u \geq 0 & \text{for every } u \in V, \\
x_u + x_v + x_w \geq 2 & \text{for every triangle } \{u, v, w\} \text{ in } G.
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Integrality gap:

Remains 2.

Gap examples:

Triangle-free graphs with small independence number.

Sherali-Adams and Lasserre/Sums-of-Squares Hierarchies

Hierarchy:

Systematic ways of
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Given a polytope:

$$P = \{x \in \mathbb{R}^n : Ax \geq b\},$$
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Produce explicit nested polytopes:

$$P = P^1 \supseteq P^2 \supseteq \cdots \supseteq P^{n-1} \supseteq P^n = P^{\mathbb{Z}}$$

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$$Q_0 + \sum_{j=1}^{m} L_j Q_j + \sum_{i=1}^{n} (x_i^2 - x_i) Q_i = L \geq 0$$

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and

$$\deg(Q_0), \deg(L_j Q_j), \deg((x_i^2 - x_i) Q_i) \leq k.$$

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Then:

$$P^k = \{x \in \mathbb{R}^n : L(x) \geq 0 \text{ for each produced } L \geq 0\}$$

P^k: Sherali-Adams (SA) Hierarchy

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produce all linear inequalities of the form

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where

$$Q_i = \sum_{\ell \in J} c_\ell \prod_{i \in A_\ell} x_i \prod_{i \in B_\ell} (1 - x_i) \quad \text{with} \quad c_\ell \geq 0$$

and

$$\deg(Q_0), \deg(L_j Q_j), \deg((x_i^2 - x_i) Q_i) \leq k.$$

Then:

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Example: triangles in P^3

For each triangle $\{u, v, w\}$ in G:

$$
\begin{aligned}
&Q_0 + \\
&(x_u + x_v - 1)Q_1 + \\
&(x_u + x_w - 1)Q_2 + \\
&(x_v + x_w - 1)Q_3 + \\
&(x_u^2 - x_u)Q_4 + \\
&(x_v^2 - x_v)Q_5 + \\
&(x_w^2 - x_w)Q_6 \\
&= ? \\
&(x_u + x_v + x_w - 2).
\end{aligned}
$$

$$Q_i = a_i + b_i x_u + c_i x_v + d_i x_w + e_i x_u x_v + f_i x_u x_w + g_i x_v x_w + h_i x_u x_v x_w$$

Solving P^k

Lift-and-project:

- Step 1: lift from $\mathbb{R}^n$ up to $\mathbb{R}^{(n+1)^k}$ and linearize the problem
- Step 2: project from $\mathbb{R}^{(n+1)^k}$ down to $\mathbb{R}^n$

Proposition:

Optimization of linear functions over P^k
can be solved in time† $m^{O(1)}n^{O(k)}$.

Proof:

1. for SA-P^k: by linear programming
2. for SOS-P^k: by semidefinite programming

An Important Open Problem

Define

$\mathrm{sa}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of SA-P^k
$\mathrm{sos}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of SOS-P^k

An Important Open Problem

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$\mathrm{sa}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of SA-P^k
$\mathrm{sos}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of SOS-P^k

Open problem:

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{sos}^4\mathrm{fvc}(G)} \overset{?}{<} 2$$

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Gap examples:

Frankl-Rödl Graphs: $\mathrm{FR}_\gamma^n = (\mathbb{F}_2^n, \{\{x, y\} : x + y \in A_\gamma^n\})$.

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Frankl-Rödl Graphs: $\mathrm{FR}^n_\gamma = (\mathbb{F}_2^n, \{\{x, y\} : x + y \in A^n_\gamma\})$.

[Dinur, Safra, Khot, Regev, Kleinberg, Charikar, Hatami, Magen, Georgiou, Lovasz, Arora, Alekhnovich, Pitassi; 2000's]

Part II

COUNTING LOGIC

Bounded-Variable Logics

First-order logic of graphs:

$E(x, y)$	:	x and y are joined by an edge
$x = y$	:	x and y denote the same vertex
$\neg\phi$	:	negation of ϕ holds
$\phi \wedge \psi$	:	both ϕ and ψ hold
$\exists x(\phi)$	:	there exists a vertex x that satisfies ϕ

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First-order logic with k variables (or width k) :

L^k: collection of formulas for which
all subformulas have at most k free variables.

Example

Paths:

$$
\begin{aligned}
P_1(x,y) &:= E(x,y) \\
P_2(x,y) &:= \exists z_1(E(x,z_1) \wedge P_1(z_1,y)) \\
P_3(x,y) &:= \exists z_2(E(x,z_2) \wedge P_2(z_2,y)) \\
&\vdots \\
P_{i+1}(x,y) &:= \exists z_i(E(x,z_i) \wedge P_i(z_i,y)) \\
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Bipartiteness of n-vertex graphs:

$$\forall x (\neg P_3(x,x) \wedge \neg P_5(x,x) \wedge \cdots \wedge \neg P_{2\lceil n/2 \rceil - 1}(x,x)).$$

Counting quantifiers

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Counting logic with k variables (or counting width k):

C^k: collection of formulas with counting quantifiers with all subformulas with at most k free variables.

C^k-**equivalence**:

$G \equiv_k^C H$: G and H satisfy the same sentences of C^k.

Combinatorial characterization of C^2-equivalence

Color-refinement:

1. color each vertex black,
2. color each vertex by number of neighbors in each color-class,
3. repeat 2 until color-classes don't split any more.

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Theorem [Immerman and Lander]

$$G \equiv_2^C H \text{ if and only if } G \equiv^R H$$

LP characterization of color-refinement

Isomorphisms:

1. $G \cong H$,
2. there exists permutation matrix P such that $P^{\mathrm{T}} G P = H$,
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LP relaxation of $\cong$:

$G \equiv^F H$: there exists doubly stochastic S such that $GS = SH$.

$$\begin{aligned} \mathrm{iso}(G, H) \quad : \quad & GS = SH \\ & Se = e^{\mathrm{T}} S = e \\ & S \geq 0. \end{aligned}$$

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Theorem [Tinhofer]

$$G \equiv^R H \text{ if and only if } G \equiv^F H.$$

Higher levels of SA Hierarchy

SA-levels of fractional isomorphism:

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Theorem [AA and Maneva 2013]:

$$G \equiv_k^{\mathrm{SA}} H \Longrightarrow G \equiv_k^{C} H \Longrightarrow G \equiv_{k-1}^{\mathrm{SA}} H.$$

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Moreover:

1. This interleaving is strict for $k > 2$ [Grohe-Otto 2015]
2. A combined LP characterizes $\equiv_k^C$ exactly [Grohe-Otto 2015]
3. Alternative (and independent) formulation by [Malkin 2014]

Higher Levels of SOS Hierarchy

SA and SOS-levels of fractional isomorphism:

1. $G \equiv_k^{\mathrm{SA}} H$: the degree-k SA level of $\mathrm{iso}(G, H)$ is feasible.
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Theorem [AA and Ochremiak 2018]: There exists $c > 1$ such that:

$$G \equiv_{ck}^{\mathrm{SA}} H \Longrightarrow G \equiv_k^{\mathrm{SOS}} H \Longrightarrow G \equiv_k^{\mathrm{SA}} H.$$

Part III

APPLICATIONS

Local LPs (and SDPs)

Basic k-local LPs:

1. one variable $x_{\mathbf{u}}$ for each k-tuple $\mathbf{u} \in V^k$,
2. one inequality $\sum_{\mathbf{u} \in V^k} a_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}} \geq b_{\mathbf{v}}$ for every k-tuple $\mathbf{v} \in V^k$,
3. coefficients $a_{\mathbf{u},\mathbf{v}}$ depend only on the type $\mathrm{atp}_G(\mathbf{u}, \mathbf{v})$,
4. coefficients $b_{\mathbf{v}}$ depend only on the type $\mathrm{atp}_G(\mathbf{v})$.

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k-local LP:

Union of basic k-local LPs
with coefficients $a_{t(\mathbf{x},\mathbf{y})}$ and $b_{t(\mathbf{y})}$ indexed
by isomorphism types $t(\mathbf{x}, \mathbf{y})$ and $t(\mathbf{y})$.

Example 1: fractional vertex cover

Fractional vertex cover: Given a graph $G = (V, E)$

$$\begin{aligned} &\textstyle\sum_{u \in V} x_u \leq W \\ &x_u + x_v \geq 1 \quad \text{for every } (u, v) \in E, \\ &x_u \geq 0 \qquad\qquad \text{for every } u \in V. \end{aligned}$$

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1. Objective function: basic 1-local LP
2. Edge constraint: basic 2-local LP
3. Positive constraint: basic 1-local LP

Example 2: fractional matching polytope

Fractional matching polytope: Given a graph $G = (V, E)$

$$
\begin{aligned}
&\textstyle\sum_{uv \in E} x_{uv} \geq W && \\
&x_{uv} = x_{vu} && \text{for every } u, v \in V \\
&\textstyle\sum_{v \in V} x_{uv} \leq 1 && \text{for every } u \in V \\
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1. Objective function: basic 2-local LP
2. Symmetry constraint: two basic 2-local LPs
3. Degree-at-most-one constraint: basic 2-local LP

Example 3: metric polytope

Metric polytope: Given a graph $G = (V, E)$

$$\begin{array}{ll} \frac{1}{2}\sum_{uv\in E} x_{uv} \geq W & \\ x_{uv} = x_{vu} & \text{for every } u, v \in V \\ x_{uw} \leq x_{uv} + x_{vw} & \text{for every } u, v, w \in V \\ x_{uv} + x_{vw} + x_{uw} \leq 2 & \text{for every } u, v, w \in V \\ 0 \leq x_{uv} \leq 1 & \text{for every } u, v \in V \end{array}$$

1. Objective function: basic 2-local LP
2. Symmetry constraint: two basic 2-local LPs
3. Triangle inequality: basic 3-local LP
4. Perimetric inequality: basic 3-local LP
5. Unit cube constraint: two basic 2-local LPs

Preservation of local LPs and SDPs

Theorem Let P be a k-local LP or SDP.

1. LP: If $G \equiv_k^C H$, then $P(G)$ is feasible iff $P(H)$ is feasible.
2. SDP: If $G \equiv_{ck}^C H$, then $P(G)$ is feasible iff $P(H)$ is feasible.

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'Just do it' proof for LP:

1. Let $\{x_{\mathbf{u}}\}$ be a feasible solution for $P(G)$.
2. Let $\{X_{\mathbf{u},\mathbf{v}}\}$ be a feasible solution for $\mathrm{sa}^k\mathrm{iso}(G, H)$.
3. Define:
$$y_{\mathbf{v}} := \sum_{\mathbf{u} \in G^k} X_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}}.$$
4. Check that $\{y_{\mathbf{v}}\}$ is a feasible solution for $P(H)$.

More examples of local LPs

More examples:

1. maximum flows (2-local)
2. if P is r-local LP, then sa^k-P is rk-local LP.
3. if P is r-local LP, then sos^k-P is rk-local SDP.

Back to integrality gaps for vertex cover

Goal:

For large k and every $\epsilon > 0$ find graphs G and H such that

1. $G \equiv^{C}_{c \cdot 2k} H$
2. $\mathrm{vc}(G) \geq (2 - \epsilon)\mathrm{vc}(H)$

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It would follow that:

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Proof:

$$\begin{aligned}
\mathrm{vc}(G) &\geq (2-\epsilon)\mathrm{vc}(H) && \text{by 2.} \\
&\geq (2-\epsilon)\mathrm{sos}^k\mathrm{fvc}(H) && \text{obvious} \\
&\geq (2-\epsilon)\mathrm{sos}^k\mathrm{fvc}(G) && \text{by 1. and 2-locality}
\end{aligned}$$

GOAL

For large k and every $\epsilon > 0$ find graphs G and H such that

1. $G \equiv^{C}_{c \cdot 2k} H$
2. $\mathrm{vc}(G) \geq (2 - \epsilon)\mathrm{vc}(H)$

A weak (easy) case: $k = 1$ with gap $= 2$

Choose:

$G =$ any d-regular expander graph (i.e., $\lambda_2(G) \ll \lambda_1(G)$),
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Then:

$\mathrm{vc}(G) = (1 - \epsilon)n$	by expansion
$\mathrm{vc}(H) = n/2$	by bipartition
$G \equiv^R H$	by regularity
$G \equiv_2^C H$	by Tinhofer's Theorem

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Tight in two ways:

$G \not\equiv_3^C H$	bipartiteness is C^3-definable,
$G \equiv_2^C H \Longrightarrow \mathrm{vc}(G) \leq 2\mathrm{vc}(H)$	[AA-Dawar 2018]

A different weak (harder) case: $k = \Omega(n)$ but gap $= 1.08$

Theorem [AA-Dawar 2018]

There exist graphs G_n and H_n such that

1. $G_n \equiv^{C}_{\Omega(n)} H_n$
2. $\mathrm{vc}(G_n) \geq 1.08 \cdot \mathrm{vc}(H_n)$

Part IV

PROOF INGREDIENTS

1/3: Locally consistent systems of linear equations

Ingredient 1: A linear system $Ax = b$ over $\mathbb{F}_2$ where:

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Probabilistic construction:

1. set $m = cn$ for a large constant $c = c(\epsilon)$
2. choose three ones uniformly at random in each row of A
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Half-deterministic construction:

1. set $m = cn$ for a large constrant $c = c(\epsilon)$
2. let A be incidence matrix of bipartite expander
3. choose b uniformly at random in $\mathbb{F}_2^n$.

2/3: Indistinguishable systems of linear equations

Ingredient 2: A pair of linear systems S_0 and S_1 over $\mathbb{F}_2$ where:

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3. some solution solution exists for S_1

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Construction of S_0:

1. start with $Ax = b$ from previous section
2. duplicate each variable $x \mapsto (x^{(0)}, x^{(1)})$
3. replace each equation $x_i + x_j + x_k = b$ by 8 equations

$$x_i^{(u)} + x_j^{(v)} + x_k^{(w)} = b + u + v + w$$

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Construction of S_1:

1. same but start with $Ax = 0$ (the homogeneous system)

3/3: Reduction to vertex cover

Ingredient 3: A pair of graphs G_0 and G_1 where:

1. $G_0 \equiv^{C}_{\Omega(n)} G_1$
2. $\mathrm{vc}(G_0) \geq 26m$
3. $\mathrm{vc}(G_1) \leq 24m$

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3. $\mathrm{vc}(G_1) \leq 24m$

Construction:

a standard reduction from $\mathbb{F}_2$-SAT to vertex cover

Open Problem 1

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{sos}^4\mathrm{fvc}(G)} > 1.36?$$

Open Problem 2

find strongly regular graphs G and H with same parameters so that $\mathrm{vc}(G) \geq (2 - \epsilon)\mathrm{vc}(H)$.

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