On Continuous and Combinatorial Relaxations of Graph Isomorphism

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Based on joint work with
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Indistinguishability
Overview:

1. Iterated degree sequences and Weisfeiler-Lehman algorithm
2. Fractional isomorphisms and Sherali-Adams relaxations
3. Transfer Lemma
4. Indistinguishability in counting logics
5. Applications
Part I

ITERATED DEGREE SEQUENCES
Iterated degree sequences

Let $G = (V, E)$ be a graph.
Use $u$ to denote a vertex, and $N_G(u)$ for its neighborhood.

**Start at the degree sequence:**

$$d_1(u) := |N_G(u)|,$$
$$d_1(G) := \{ \{ d_1(u) : u \in V \} \}.$$ 

**Iterate:**

$$d_{i+1}(u) := \{ \{ d_i(v) : v \in N_G(u) \} \},$$
$$d_{i+1}(G) := \{ \{ d_{i+1}(u) : u \in V \} \}.$$ 

**Take the limit:**

$$D(G) := (d_1(G), d_2(G), d_3(G), \ldots).$$
Indistinguishability by iterated degree sequences

Definition:

\[ G \cong_D H \text{ iff } D(G) = D(H). \]
Indistinguishability by iterated degree sequences

$\cong_D$ is strong...

**Theorem** [Babai-Erdös-Selkow 80]:

Let $G = G(n, 1/2)$ be drawn randomly. Then, a.s. as $n \to \infty$, for every $H$ with $n$ vertices we have $G \cong_D H$ iff $G \cong H$. 
Indistinguishability by iterated degree sequences

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Let \( G = G(n, 1/2) \) be drawn randomly. Then, a.s. as \( n \to \infty \), for every \( H \) with \( n \) vertices we have \( G \cong_D H \) iff \( G \cong H \).

But also **weak...**

**Fact** [Obvious]:

If \( G \) and \( H \) are both \( d \)-regular, then \( G \cong_D H \).
Types of $k$-tuples

For a $k$-tuple of vertices $\overline{u} = (u_1, \ldots, u_k) \in V^k$,

Define:

$$tp_G(\overline{u}) = \text{“complete information about adjacencies, non-adjacencies, equalities and non-equalities between the components } u_1, \ldots, u_k \text{”}.$$ 

Example:

$$tp_G(u_1, u_2, u_3) = \{ \overline{E}(1, 1), E(1, 2), E(1, 3), \overline{E}(2, 1), \overline{E}(2, 2), \overline{E}(3, 2), E(3, 1), \overline{E}(3, 2), \overline{E}(3, 3), 1 \neq 2, 1 \neq 3, 2 = 3 \}$$
Start at the type sequence:

\[ \ell_0(\overline{u}) := \text{tp}_G(\overline{u}), \]
\[ \ell_0(G) := \{ \{ \ell_0(\overline{u}) : \overline{u} \in V^k \} \}. \]

Iterate:

\[ \ell_{i+1}(\overline{u}) := \{ \{ \text{tp}_G(\overline{u}v), \ell_i(\overline{u}[1/v]), \ldots, \ell_i(\overline{u}[k/v]) : v \in V \} \}, \]
\[ \ell_{i+1}(G) := \{ \{ \ell_{i+1}(\overline{u}) : \overline{u} \in V^k \} \}. \]

Take the limit:

\[ D^k(G) := (\ell_0(G), \ell_1(G), \ldots). \]
Definition:

\[ G \cong^{k}_{\text{WL}} H \text{ iff } D^{k}(G) = D^{k}(H). \]
Indistinguishability by $k$-dim WL

$\cong^k_{WL}$ is strong...

At least as strong as vertex-refinement:

$$G \not\cong_D H \implies G \not\cong^1_{WL} H$$

**Theorem** [Kucera 87]:

Let $G = G_{\text{reg}}(n, d)$ be drawn randomly. Then, a.s. as $n \to \infty$, for every $H$ with $n$ vertices we have $G \cong^2_{WL} H$ iff $G \cong H$. 

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**Relevant note:**

$\cong^k_{\text{WL}}$ is decidable in time $n^{O(k)}$. 
Is $k$-dim WL weak at all?

**Truth is:**

For years no two $\cong_{WL}^{37}$-indistinguishable graphs were known...
It was even *conjectured* that no such graphs existed...
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Theorem [Cai-Fürer-Immerman 92]:

There exists explicitly defined graphs $G_n$ and $H_n$, with $n$ vertices each and maximum degree 3, such that

$$G_n \cong_{\text{WL}}^{\Omega(n)} H_n \quad \text{yet} \quad G_n \not\cong H_n.$$

Note:

Reasoning about $\cong_{\text{WL}}^{k}$ requires an excursion into finite model theory (more on this later).
CFI-construction

1. Start with a 3-regular graph $G$ without $\Omega(n)$-separators.
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2. Replace each vertex by gadget:
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2. Replace each vertex by gadget:

3. Let $G_n$ be the result and let $H_n = G_n + \text{“one flip”}$.
Part II

SHERALI-ADAMS RELAXATIONS
## Adjacency matrices

Let $G = (V^G, E^G)$ and $H = (V^H, E^H)$ be graphs.

Say $V^G = V^H = \{1, \ldots, n\}$.

Let $A$ and $B$ be their adjacency matrices.

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]
Permutation matrices and isomorphisms

A permutation matrix $P$ is a real matrix such that

$$
\sum_{j=1}^{n} P_{ij} = 1 \quad \text{for every } i \in \{1, \ldots, n\},
$$

$$
\sum_{i=1}^{n} P_{ij} = 1 \quad \text{for every } j \in \{1, \ldots, n\},
$$

$$
P_{ij} \in \{0, 1\} \quad \text{for every } i, j \in \{1, \ldots, n\}.
$$

**Properties:**

- $P^T P = I$,
- $A \mapsto AP$: permutes the columns of $A$,
- $A \mapsto P^T A$: permutes the rows of $A$,
- $A \mapsto P^T AP$: permutes the vertices.

**Fact:** The following are equivalent:

1. $G \cong H$,
2. there exists $P \in \mathcal{P}_n$ such that $P^T AP = B$,
3. there exists $P \in \mathcal{P}_n$ such that $AP = PB$. 
A doubly stochastic matrix $S$ is a real matrix such that:

\[
\sum_{j=1}^{n} S_{ij} = 1 \quad \text{for every } i \in \{1, \ldots, n\},
\]
\[
\sum_{i=1}^{n} S_{ij} = 1 \quad \text{for every } j \in \{1, \ldots, n\},
\]
\[
S_{ij} \geq 0 \quad \text{for every } i, j \in \{1, \ldots, n\}.
\]

Relaxation of isomorphism:

- Replace “there exists $P \in \mathcal{P}_n$ such that $AP = PB$”
- by this “there exists $S \in S_n$ such that $AS = SB$”.

In other words, let $I(G, H)$ be the LP for $S_n$ plus

\[
\sum_{i=1}^{n} A_{ui} S_{iv} = \sum_{j=1}^{n} S_{uj} B_{jv}
\]

for every $u, v \in V^G \times V^H$. 
Indistinguishability by fractional isomorphisms

Definition:

\[ G \cong_F H \text{ iff } I(G, H) \neq \emptyset. \]
Indistinguishability by fractional isomorphisms

Suppose \( G \cong_F H \). Then:

- \( |E^G| = |E^H| \),
- actually \( d_1(G) = d_1(H) \),
- and even \( D(G) = D(H) \).
Indistinguishability by fractional isomorphisms

Suppose $G \cong_F H$. Then:

- $|E^G| = |E^H|$, 
- actually $d_1(G) = d_1(H)$, 
- and even $D(G) = D(H)$.

Indeed:

**Theorem** [Ramana-Scheinerman-Ullman 94]

\[ G \cong_F H \iff G \cong_D H. \]
Let

\[ P = \{ x \in \mathbb{R}^n : Ax \geq b \}, \]

\[ P^Z = \text{convexhull}\{ x \in \{0, 1\}^n : Ax \geq b \}. \]

The Sherali-Adams levels are nested polytopes:

\[ P = P^0 \supset P^1 \supset P^2 \supset \cdots \supset P^n = P^Z \]

and the SA-rank of \( P \) is:

\[ \min\{ k : P^k = P^Z \}. \]
Definition of $P^k$ in four steps

Let

$$P^k = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} a_1^T x \geq b_1 \\ \vdots \\ a_m^T x \geq b_m \end{bmatrix} \right\}.$$ 

be the LP.
Definition of $P^k$ in four steps

Step 1: Multiply each $a_i^T x \geq b_i$ by all multipliers of the form

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$$

for $I, J \subseteq [n]$, $|I \cup J| \leq k - 1$, $I \cap J = \emptyset$. 

Definition of $P^k$ in four steps

**Step 1**: Multiply each $a_i^T x \geq b_i$ by all multipliers of the form

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for $I, J \subseteq [n]$, $|I \cup J| \leq k - 1$, $I \cap J = \emptyset$.

Step 1 leaves an equivalent system of polynomials of degree $k$. 
Definition of $P^k$ in four steps

**Step 2:** Expand the products and replace each square $x_i^2$ by $x_i$. 
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Step 2 leaves a system of multi-linear polynomials of degree $k$. This is the **integrality** step: valid on $\{0, 1\}^n$ only.
Definition of $P^k$ in four steps

**Step 3:** Linearize each monomial $\prod_{i \in I} x_i$ by introducing a new variable $y_I$. 
Definition of $P^k$ in four steps

**Step 3**: Linearize each monomial $\prod_{i \in I} x_i$ by introducing a new variable $y_I$.

Step 3 leaves a linear program $Q^k$ on the $y_I$-variables in $\mathbb{R}^{n^k}$. This is the relaxation step.
Definition of $P^k$ in four steps

Step 4: Define

$$P^k := \{ x \in \mathbb{R}^n : \exists y \in Q^k \text{ s.t. } y\{i\} = x_i \text{ for every } i \}.$$
Definition of $P^k$ in four steps

**Step 4:** Define

$$P^k := \{ x \in \mathbb{R}^n : \exists y \in Q^k \text{ s.t. } y\{i\} = x_i \text{ for every } i \}.$$  

Step 4 takes us back to $\mathbb{R}^n$. It’s the *projection* step: from $\mathbb{R}^{n_k}$ to $\mathbb{R}^n$.  


Note:

The polytope $P^k$ is definable by an LP on $n^k$ variables and $m \cdot n^k$ inequalities.

Therefore:

Feasibility and optimization of linear functions over $P^k$ can be solved in time $m^{O(1)} n^{O(k)}$. 
Definition:

\[ G \cong^k_{SA} H \text{ iff } I(G, H)^k \neq \emptyset. \]
Part III

TRANSFER LEMMA
Statement of the transfer lemma

Transfer Lemma:

\[ G \cong_{WL}^k H \implies G \cong_{SA}^{k-1} H \implies G \cong_{WL}^{k-1} H. \]

Interpretation:

A geometric concept is captured by purely combinatorial means. A combinatorial concept is captured by purely geometric means.
Proof of the transfer lemma

Intermediate notions of indistinguishability:

\[ G \cong_{\text{WL}}^k H \Rightarrow G \cong_{\text{C}}^k H \Rightarrow G \cong_{\text{CS}}^{k-1} H \Rightarrow G \cong_{\text{EP}}^{k-1} H \Rightarrow G \cong_{\text{SA}}^{k-1} H \]

and

\[ G \cong_{\text{SA}}^{k-1} H \Rightarrow G \cong_{\text{C}}^{k-1} H \Rightarrow G \cong_{\text{WL}}^{k-1} H. \]
Proof of the transfer lemma

Intermediate notions of indistinguishability:

\[ G \cong_{WL}^k H \Rightarrow G \cong_{C}^k H \Rightarrow G \cong_{CS}^{k-1} H \Rightarrow G \cong_{EP}^{k-1} H \Rightarrow G \cong_{SA}^{k-1} H \]

and

\[ G \cong_{SA}^{k-1} H \Rightarrow G \cong_{C}^{k-1} H \Rightarrow G \cong_{WL}^{k-1} H. \]

Here:

\[ \cong_{C}^k \] is indistinguishability by properties definable in first-order logic with counting quantifiers and width \( k \).
Part IV

COUNTING LOGICS
Indistinguishability
Counting quantifiers

**Counting witnesses:**

\[ \exists \geq i x(\phi(x)) : \text{there are at least } i \text{ vertices } x \text{ that satisfy } \phi(x). \]

**Example:**

\[ \psi_d(x) := \exists \geq d y(E(x, y)) \land \neg \exists^{d+1} y(E(x, y)), \]
\[ \phi := \neg \exists \geq 1 x(\neg \psi_d(x)). \]

**Note:**

We used only **two** first-order variables (\(x\) and \(y\)) where \(d + 1\) are required in pure first-order logic.
Bounded width formulas

Example: First paths

\[ P_1(x, y) := E(x, y) \]
\[ P_2(x, y) := \exists z_1 (E(x, z_1) \land P_1(z_1, y)) \]
\[ P_3(x, y) := \exists z_2 (E(x, z_2) \land P_2(z_2, y)) \]
\[ \vdots \]
\[ P_{i+1}(x, y) := \exists z_i (E(x, z_i) \land P_i(z_i, y)). \]

and then

\[ \forall x (\neg P_3(x, x) \land \neg P_5(x, x) \land \cdots \land \neg P_{2\lceil n/2 \rceil-1}(x, x)). \]

Counting logic with \( k \) variables:

\( C^k \): collection of formulas for which all subformulas have at most \( k \) free variables.
Indistinguishability by $C^k$

**Definition:**

$G \sim_C^k H$ iff for every $\phi \in C^k$ we have $G \models \phi \iff H \models \phi$. 
Pebble game (without counting moves)

Forced win for Spoiler.
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Forced win for Spoiler.
A winning strategy for the Duplicator in $G \simeq^k H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from $G$ to $H$ such that for every $f \in \mathcal{F}$ we have:
Systems with the back-and-forth properties

A winning strategy for the Duplicator in $G \cong^k \forall H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from $G$ to $H$ such that for every $f \in \mathcal{F}$ we have:

1. (bounded) $|\text{Dom}(f)| \leq k$, 


Systems with the back-and-forth properties

A winning strategy for the Duplicator in \( G \cong^{k}_C H \) is a non-empty collection \( \mathcal{F} \) of partial isomorphisms from \( G \) to \( H \) such that for every \( f \in \mathcal{F} \) we have:

1. (bounded) \( |\text{Dom}(f)| \leq k \),
2. (subfunction) For every \( g \subseteq f \) we have \( g \in \mathcal{F} \),
Systems with the back-and-forth properties

A **winning strategy** for the Duplicator in $G \cong^k \mathcal{O} H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from $G$ to $H$ such that for every $f \in \mathcal{F}$ we have:

1. (bounded) $|\text{Dom}(f)| \leq k$,
2. (subfunction) For every $g \subseteq f$ we have $g \in \mathcal{F}$,
3. (back) If $|\text{Dom}(f)| < k$ then:
   - for every $X \subseteq V_G$ there exists $Y \subseteq V_H$ with $|Y| = |X|$ s.t.
   - for every $v \in Y$ there exists $u \in X$ with $f \cup \{(u, v)\} \in \mathcal{F}$,
A **winning strategy** for the Duplicator in $G \cong^k \sim H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from $G$ to $H$ such that for every $f \in \mathcal{F}$ we have:

1. (bounded) $|\text{Dom}(f)| \leq k$,
2. (subfunction) For every $g \subset f$ we have $g \in \mathcal{F}$,
3. (back) If $|\text{Dom}(f)| < k$ then:
   - for every $X \subseteq V_G$ there exists $Y \subseteq V_H$ with $|Y| = |X|$ s.t.
     - for every $v \in Y$ there exists $u \in X$ with $f \cup \{(u, v)\} \in \mathcal{F}$,
4. (forth) If $|\text{Dom}(f)| < k$ then:
   - for every $Y \subseteq V_H$ there exists $X \subseteq V_G$ with $|X| = |Y|$ s.t.
     - for every $u \in X$ there exists $v \in Y$ with $f \cup \{(u, v)\} \in \mathcal{F}$.
Theorem [Immerman-Lander 90, Cai-Fürer-Immerman 92]

\[ G \cong^k_{\text{WL}} H \iff G \cong^{k+1}_{C} H. \]

Relevant note: From its definition, it is not even obvious that \( G \cong^k_{C} H \) is decidable in time \( n^{O(k)} \).
Wanted:

\[ G \cong_{SA}^k H \implies G \cong_{C}^k H \]

**Ingredient 1:**

Birkhoff decomposition theorem: every doubly stochastic matrix is a **convex combination** of permutation matrices.

**Ingredient 2:**

Permutations preserve sizes of sets.
From systems with B&F to feasible solutions

Wanted:

\[ G \cong^k_C H \implies G \cong^{k-1}_{SA} H \]

**Ingredient 1:**

A sliding game to account for \( AS = SB \);
here is where the \(-1\) is lost.

**Ingredient 2:**

Normalizing winning strategies into uniform ones.
Part V

APPLICATIONS (or what to do of this?)
Isomorphism testing for special graphs

**Theorem** [Immerman-Lander 90, Grohe 98, ...]

1. If $G$ is a tree, then $G \cong^2_C H$ iff $G \cong H$, for every $H$.
2. If $G$ is planar, then $G \cong^{15}_C H$ iff $G \cong H$, for every $H$.
3. ... 

**Corollary**

For all such graph classes, an explicit and poly-size LP solves graph isomorphism.
Consider the standard LP-relaxation of vertex cover:

\[
\begin{align*}
\text{minimize} \quad & \sum_{u \in V} x_u \\
\text{subject to} \quad & x_u + x_v \geq 1 \quad \text{for every} \ (u, v) \in E, \\
& x_u \geq 0 \quad \text{for every} \ u \in V.
\end{align*}
\]
SA-rank lower bounds

Consider the standard LP-relaxation of vertex cover:

\[
\begin{align*}
\text{minimize} & \quad \sum_{u \in V} x_u \\
\text{subject to} & \quad x_u + x_v \geq 1 \quad \text{for every } (u, v) \in E, \\
& \quad x_u \geq 0 \quad \text{for every } u \in V.
\end{align*}
\]

We expect that the inequality

\[
\sum_{u \in V} x_u \geq \text{vc}(G) \tag{1}
\]

will not, in general, be valid over $P^k(G)$ for any $k = O(1)$. 
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\sum_{u \in V} x_u \geq \text{vc}(G) \tag{1}
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will not, in general, be valid over \( P^k(G) \) for any \( k = O(1) \).

Indeed:

**Theorem** [exercise, also follows Schoenebeck 08]

There exist graphs \( G \) for which (2) is not valid over \( P^{\Omega(n)}(G) \).
Sketch:

1. Start with the $n$-vertex CFI graphs $G \cong \Omega(n)$ $H$ yet $G \not\cong H$.
2. In particular $(G, G) \cong \Omega(n)$ $(G, H)$ yet $G \cong G$ and $G \not\cong H$.
3. Apply the reduction from graph isomorphism to vertex cover.
4. Get graphs $A \cong \Omega(n)$ $B$ with $vc(A) \neq vc(B)$.
5. Apply transfer lemma and get $A \cong \Omega(n)$ $B$.

Final step:

$$A \cong^{2k}_{SA} B \implies opt(P^k(A)) = opt(P^k(B)).$$
Consider the standard LP-relaxation of \textit{max-cut}:

\[
\text{maximize } \frac{1}{2} \sum_{uv \in E} x_{uv} \\
\text{subject to }
\begin{align*}
    x_{uv} &= x_{vu} \\
    x_{uw} &\leq x_{uv} + x_{vw} \\
    x_{uv} + x_{vw} + x_{wu} &\leq 2 \\
    0 &\leq x_{uv} \leq 1
\end{align*}
\]
SA-rank lower bounds

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&\quad x_{uw} \leq x_{uv} + x_{vw} \\
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&\quad 0 \leq x_{uv} \leq 1
\end{align*}
\]

We expect that the inequality

\[
\sum_{u \in V} x_u \leq mc(G) \tag{2}
\]

will not, in general, be valid over \(P^k(G)\) for any \(k = O(1)\).
Consider the standard LP-relaxation of \textit{max-cut}:

\[
\text{maximize } \frac{1}{2} \sum_{uv \in E} x_{uv} \\
\text{subject to}
\]

\[
x_{uv} = x_{vu} \\
x_{uw} \leq x_{uv} + x_{vw} \\
x_{uv} + x_{vw} + x_{wu} \leq 2 \\
0 \leq x_{uv} \leq 1
\]

We expect that the inequality

\[
\sum_{u \in V} x_u \leq \text{mc}(G) \tag{2}
\]

will not, in general, be valid over $P^k(G)$ for any $k = O(1)$. Indeed:

**Theorem** [follows from Schoenebeck 08]

There exist graphs $G$ for which (2) is not valid over $P^{\Omega(n)}(G)$.  

**New proof**

**Sketch:**

1. Start with the $n$-vertex CFI graphs $G \cong_{\Omega(n)} H$ yet $G \not\cong H$.  
2. In particular $(G, G) \cong_{\Omega(n)} C(G, H)$ yet $G \cong G$ and $G \not\cong H$.  
3. Apply the **reduction** from graph isomorphism to max-cut.  
4. Get graphs $A \cong_{\Omega(n)} C B$ with $mc(A) \neq mc(B)$.  
5. Apply **transfer lemma** and get $A \cong_{\Omega(n)} SA B$.  

**Final step:**

$$A \cong_{3k}^{SA} B \iff \text{opt}(P^k(A)) = \text{opt}(P^k(B)).$$
Local LPs

**Basic $k$-local LPs:**

1. one variable $x_u$ for each $k$-tuple $u \in V^k$,
2. one inequality $\sum_{u \in V^k} a_{u,v} \cdot x_u \geq b_v$ for every $k$-tuple $v \in V^k$,
3. coefficients $a_{u,v}$ depend only on the type $t_G(u,v)$,
4. coefficients $b_v$ depend only on the type $t_G(v)$.

**Generic $k$-local LPs:**

Unions of generic basic $k$-local LPs (with coefficients given as a function of the types).

**Instantiation of generic $k$-local LPs:**

Let $P$ is a generic $k$-local LP. Then $P(G)$ is the LP associated to $G$. 
Recall the metric polytope:

\[
\frac{1}{2} \sum_{uv \in E} x_{uv} \geq W \\
x_{uv} = x_{vu} \\
x_{uw} \leq x_{uv} + x_{vw} \\
x_{uv} + x_{vw} + x_{uw} \leq 2 \\
0 \leq x_{uv} \leq 1
\]

1. Objective function: basic 2-local LP
2. Symmetry constraint: two basic 2-local LPs
3. Triangle inequality: basic 3-local LP
4. Perimetric inequality: basic 3-local LP
5. Unit cube constraint: two basic 2-local LPs
**Theorem:** Let $P$ be a generic $k$-local LP.

If $G \cong^k_{SA} H$, then $P(G)$ is feasible iff $P(H)$ is feasible.

'**Just do it**' proof:

1. Let $\{x_u\}$ be a feasible solution for $P(G)$.
2. Let $\{X_{u,v}\}$ be a feasible solution for $I(G, H)^k$.
3. Define:
   \[
   y_v := \sum_{u \in G^k} X_{u,v} \cdot x_u.
   \]
4. Check that $\{y_v\}$ is a feasible solution for $P(H)$.
More examples of local LPs

More examples:

1. maximum flows (2-local)
2. matchings on bipartite graphs (2-local)
3. relaxation of max-cut via the metric polytope (3-local)
4. relaxation of vertex cover (2-local)
5. \( r \) SA-levels of \( k \)-local LPs are \( O(kr) \)-local LPs.
Expressibility results

Consider the max-flow LP. It is 2-local. It is integral.

**Corollary**

$$G \cong^3_C H \Rightarrow \text{mf}(G) = \text{mf}(H).$$

**Corollary**

There exists a sentence in $C^3$ that, over $st$-networks with $n$ vertices, defines those whose maximum flow is at least the out-degree of the source.
Expressibility results

Consider the metric polytope again.

**Theorem** [Barahona-Majoub 86]:

If $G$ is a $K_5$ minor-free graph, then $\text{mc}(G) = \text{opt}(P(G))$.

**Corollary**

If $G$ and $H$ are $K_5$ minor-free, then $G \cong^4_C H \Rightarrow \text{mc}(G) = \text{mc}(H)$.

**Corollary**

There exists a sentence in $C^4$ that, over $K_5$ minor-free $n$-vertex graphs, defines those whose max-cut is at least $n/4$. 
Part VI

DISCUSSION AND OPEN PROBLEMS
Get new rank lower bounds from inexpressibility results?

Challenging problem:

Prove that an integrality gap of $2 - \epsilon$ resists $\Omega(n)$ SA-levels of vertex-cover.
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Prove that an integrality gap of $2 - \epsilon$ resists $\Omega(n)$ SA-levels of vertex-cover.

What would be enough?:

Find $G$ and $H$ such that:

1. $mc(G) \geq (2 - \epsilon) \cdot mc(H)$
2. $G \approx_{\frac{n}{C}} H$. 
New expressibility/inexpressibility results?

Challenging problem:

Is \textit{perfect matching} definable in $C^{O(1)}$?

(answer is YES for bipartite graphs)
New expressibility/inexpressibility results?

Challenging problem:

Is perfect matching definable in $C^{O(1)}$?
(answer is YES for bipartite graphs)

SOLVED! [Anderson-Dawar-Holm 13]:

YES even for general graphs!
TODA!