On Continuous and Combinatorial Relaxations of Graph Isomorphism

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Based on joint work with
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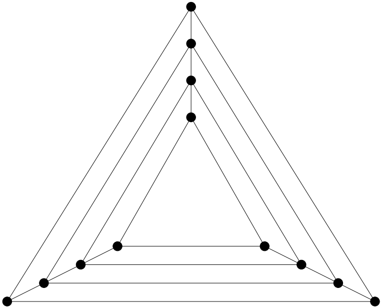
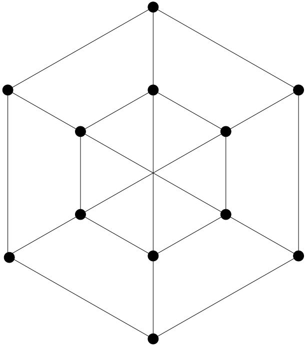
Indistinguishability





Overview of the talk

Overview:

1. Iterated degree sequences and Weisfeiler-Lehman algorithm
2. Fractional isomorphisms and Sherali-Adams relaxations
3. Transfer Lemma
4. Indistinguishability in counting logics
5. Applications

Part I

ITERATED DEGREE SEQUENCES

Iterated degree sequences

Let $G = (V, E)$ be a graph.
Use u to denote a vertex, and $N_G(u)$ for its neighborhood.

Start at the degree sequence:

$$d_1(u) := |N_G(u)|,$$
$$d_1(G) := \{\{\ d_1(u) : u \in V\ \}\}.$$

Iterate:

$$d_{i+1}(u) := \{\{\ d_i(v) : v \in N_G(u)\ \}\},$$
$$d_{i+1}(G) := \{\{\ d_{i+1}(u) : u \in V\ \}\}.$$

Take the limit:

$$D(G) := (d_1(G), d_2(G), d_3(G), \ldots).$$

Indistinguishability by iterated degree sequences

Definition:

$$G \cong_D H \text{ iff } D(G) = D(H).$$

Indistinguishability by iterated degree sequences

$\cong_D$ is strong...

Theorem [Babai-Erdös-Selkow 80]:

Let $G = G(n, 1/2)$ be drawn randomly. Then, a.s. as $n \to \infty$, for every H with n vertices we have $G \cong_D H$ iff $G \cong H$.

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But also weak...

Fact [Obvious]:

If G and H are both d-regular, then $G \cong_D H$.

Types of k-tuples

For a k-tuple of vertices $\overline{u} = (u_1, \ldots, u_k) \in V^k$,

Define:

$$\mathrm{tp}_G(\overline{u}) = \text{“complete information about adjacencies, non-adjacencies, equalities and non-equalities between the components } u_1, \ldots, u_k\text{”}.$$

Example:

$$\begin{aligned}\mathrm{tp}_G(u_1, u_2, u_3) = \{&\overline{E(1,1)}, E(1,2), E(1,3),\\ &E(2,1), \overline{E(2,2)}, \overline{E(3,2)},\\ &E(3,1), \overline{E(3,2)}, \overline{E(3,3)},\\ &1 \neq 2, 1 \neq 3, 2 = 3\}\end{aligned}$$

k-dimensional Weisfeiler-Lehman algorithm

Start at the type sequence:

$$\ell_0(\overline{u}) := \mathrm{tp}_G(\overline{u}),$$
$$\ell_0(G) := \{\{\ \ell_0(\overline{u}) : \overline{u} \in V^k\ \}\}.$$

Iterate:

$$\ell_{i+1}(\overline{u}) := \{\{\ (\mathrm{tp}_G(\overline{u}v), \ell_i(\overline{u}[1/v]), \ldots, \ell_i(\overline{u}[k/v])) : v \in V\ \}\},$$
$$\ell_{i+1}(G) := \{\{\ \ell_{i+1}(\overline{u}) : \overline{u} \in V^k\ \}\}.$$

Take the limit:

$$D^k(G) := (\ell_0(G), \ell_1(G), \ldots).$$

Indistinguishability by k-dim WL

Definition:

$$G \cong^k_{\mathrm{WL}} H \text{ iff } D^k(G) = D^k(H).$$

Indistinguishability by k-dim WL

$\cong^k_{\mathrm{WL}}$ is strong...

At least as strong as vertex-refinement:

$$G \ncong_D H \Longrightarrow G \ncong^1_{\mathrm{WL}} H$$

Theorem [Kucera 87]:

Let $G = G_{\mathrm{reg}}(n, d)$ be drawn randomly. Then, a.s. as $n \to \infty$, for every H with n vertices we have $G \cong^2_{\mathrm{WL}} H$ iff $G \cong H$.

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Relevant note:

$$\cong^k_{\mathrm{WL}} \text{ is decidable in time } n^{O(k)}.$$

Is k-dim WL weak at all?

Truth is:

For years no two $\cong_{\mathrm{WL}}^{37}$-indistinguishable graphs were known...
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Theorem [Cai-Fürer-Immerman 92]:

There exists explicitely defined graphs G_n and H_n,
with n vertices each and maximum degree 3, such that

$$G_n \cong^{\Omega(n)}_{\mathrm{WL}} H_n \quad \text{yet} \quad G_n \not\cong H_n.$$

Note:

Reasoning about $\cong^{k}_{\mathrm{WL}}$ requires an excursion
into finite model theory (more on this later).

CFI-construction

1. Start with a 3-regular graph G without $\Omega(n)$-separators.

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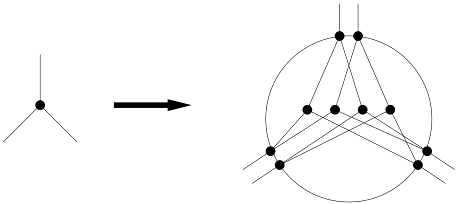

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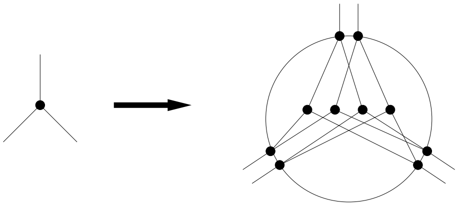
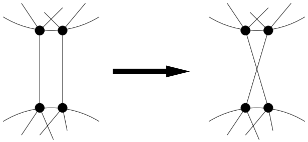


3. Let G_n be the result and let $H_n = G_n +$ "one flip".



Part II

SHERALI-ADAMS RELAXATIONS

Adjacency matrices

Let $G = (V^G, E^G)$ and $H = (V^H, E^H)$ be graphs.
Say $V^G = V^H = \{1, \ldots, n\}$.
Let A and B be their adjacency matrices.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Permutation matrices and isomorphisms

A permutation matrix P is a real matrix such that

$$\begin{array}{ll} \sum_{j=1}^{n} P_{ij} = 1 & \text{for every } i \in \{1, \ldots, n\}, \\ \sum_{i=1}^{n} P_{ij} = 1 & \text{for every } j \in \{1, \ldots, n\}, \\ P_{ij} \in \{0, 1\} & \text{for every } i, j \in \{1, \ldots, n\}. \end{array}$$

Properties:

- $P^{\mathrm{T}} P = I$,
- $A \mapsto AP$: permutes the columns of A,
- $A \mapsto P^{\mathrm{T}} A$: permutes the rows of A,
- $A \mapsto P^{\mathrm{T}} AP$: permutes the vertices.

Fact: The following are equivalent:

1. $G \cong H$,
2. there exists $P \in \mathcal{P}_n$ such that $P^{\mathrm{T}} AP = B$,
3. there exists $P \in \mathcal{P}_n$ such that $AP = PB$.

Doubly stochastic matrices and fractional isomorphisms

A doubly stochastic matrix S is a real matrix such that:

$$\begin{array}{ll} \sum_{j=1}^{n} S_{ij} = 1 & \text{for every } i \in \{1, \dots, n\}, \\ \sum_{i=1}^{n} S_{ij} = 1 & \text{for every } j \in \{1, \dots, n\}, \\ S_{ij} \geq 0 & \text{for every } i, j \in \{1, \dots, n\}. \end{array}$$

Relaxation of isomorphism:

- Replace "there exists $P \in \mathcal{P}_n$ such that $AP = PB$"
- by this "there exists $S \in \mathcal{S}_n$ such that $AS = SB$".

In other words, let $I(G, H)$ be the LP for $\mathcal{S}_n$ plus

$$\sum_{i=1}^{n} A_{ui} S_{iv} = \sum_{j=1}^{n} S_{uj} B_{jv}$$

for every $u, v \in V^G \times V^H$.

Indistinguishability by fractional isomorphisms

Definition:

$$G \cong_F H \text{ iff } I(G, H) \neq \emptyset.$$

Indistinguishability by fractional isomorphisms

Suppose $G \cong_F H$. Then:

- $|E^G| = |E^H|$,
- actually $d_1(G) = d_1(H)$,
- and even $D(G) = D(H)$.

Indistinguishability by fractional isomorphisms

Suppose $G \cong_F H$. Then:

- $|E^G| = |E^H|$,
- actually $d_1(G) = d_1(H)$,
- and even $D(G) = D(H)$.

Indeed:

Theorem [Ramana-Scheinerman-Ullman 94]

$$G \cong_F H \text{ iff } G \cong_D H.$$

Sherali-Adams relaxations

Let

$$P = \{x \in \mathbb{R}^n : Ax \geq b\},$$
$$P^{\mathbb{Z}} = \mathrm{convexhull}\{x \in \{0,1\}^n : Ax \geq b\}.$$

The Sherali-Adams levels are nested polytopes:

$$P = P^0 \supseteq P^1 \supseteq P^2 \supseteq \cdots \supseteq P^n = P^{\mathbb{Z}}$$

and the SA-rank of P is:

$$\min\{k : P^k = P^{\mathbb{Z}}\}.$$

Definition of P^k in four steps

Let

$$P = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} a_1^{\mathrm{T}} x \geq b_1 \\ \vdots \\ a_m^{\mathrm{T}} x \geq b_m \end{bmatrix} \right\}.$$

be the LP.

Definition of P^k in four steps

Step 1: Multiply each $a_i^{\mathrm{T}} x \geq b_i$ by all multipliers of the form

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$$

for $I, J \subseteq [n]$, $|I \cup J| \leq k - 1$, $I \cap J = \emptyset$.

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Step 1 leaves an equivalent system of polynomials of degree k.

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Step 2 leaves a system of multi-linear polynomials of degree k.
This is the integrality step: valid on $\{0, 1\}^n$ only.

Definition of P^k in four steps

Step 3: Linearize each monomial $\prod_{i \in I} x_i$ by introducing a new variable y_I.

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Step 3 leaves a linear program Q^k on the y_I-variables in $\mathbb{R}^{n^k}$. This is the relaxation step.

Definition of P^k in four steps

Step 4: Define

$$P^k := \{x \in \mathbb{R}^n : \exists y \in Q^k \text{ s.t. } y_{\{i\}} = x_i \text{ for every } i\}.$$

Definition of P^k in four steps

Step 4: Define

$$P^k := \{x \in \mathbb{R}^n : \exists y \in Q^k \text{ s.t. } y_{\{i\}} = x_i \text{ for every } i\}.$$

Step 4 takes us back to $\mathbb{R}^n$.
It's the projection step: from $\mathbb{R}^{n^k}$ to $\mathbb{R}^n$.

Solving P^k

Note:

The polytope P^k is definable by
an LP on n^k variables and $m \cdot n^k$ inequalities.

Therefore:

Feasibility and optimization of linear functions over P^k
can be solved in time $m^{O(1)} n^{O(k)}$.

Indistinguishability by SA-levels of fractional isomorphisms

Definition:

$$G \cong^k_{\mathrm{SA}} H \text{ iff } I(G,H)^k \neq \emptyset.$$

Part III

TRANSFER LEMMA

Statement of the transfer lemma

Transfer Lemma:

$$G \cong^{k}_{\mathrm{WL}} H \Longrightarrow G \cong^{k-1}_{\mathrm{SA}} H \Longrightarrow G \cong^{k-1}_{\mathrm{WL}} H.$$

Interpretation:

A geometric concept is captured by purely combinatorial means.
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Proof of the transfer lemma

Intermediate notions of indistinguishability:

$$G \cong_{\mathrm{WL}}^{k} H \Rightarrow G \cong_{\mathrm{C}}^{k} H \Rightarrow G \cong_{\mathrm{CS}}^{k-1} H \Rightarrow G \cong_{\mathrm{EP}}^{k-1} H \Rightarrow G \cong_{\mathrm{SA}}^{k-1} H$$

and

$$G \cong_{\mathrm{SA}}^{k-1} H \Rightarrow G \cong_{\mathrm{C}}^{k-1} H \Rightarrow G \cong_{\mathrm{WL}}^{k-1} H.$$

Proof of the transfer lemma

Intermediate notions of indistinguishability:

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and

$$G \cong^{k-1}_{\mathrm{SA}} H \Rightarrow G \cong^{k-1}_{\mathrm{C}} H \Rightarrow G \cong^{k-1}_{\mathrm{WL}} H.$$

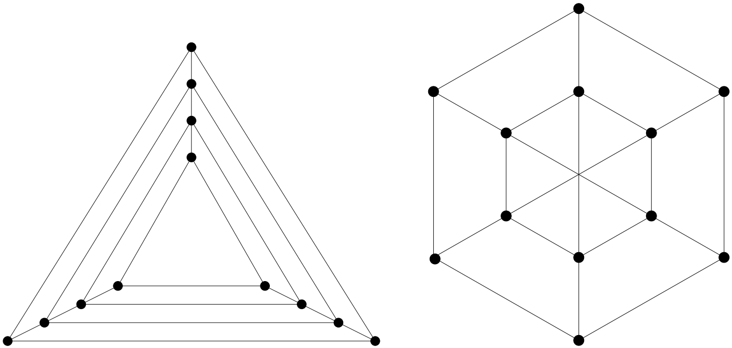
Here:

$\cong^{k}_{\mathrm{C}}$ is indistinguishability by properties definable in first-order logic with counting quantifiers and width k.

Part IV

COUNTING LOGICS

Indistinguishability



Counting quantifiers

Counting witnesses:

$\exists^{\geq i} x(\phi(x))$: there are at least i vertices x that satisfy $\phi(x)$.

Example:

$$\psi_d(x) := \exists^{\geq d} y(E(x,y)) \wedge \neg\exists^{d+1} y(E(x,y)),$$
$$\phi := \neg\exists^{\geq 1} x(\neg\psi_d(x)).$$

Note:

We used only **two** first-order variables (x and y) where $d+1$ are required in pure first-order logic.

Bounded width formulas

Example: First paths

$$
\begin{aligned}
P_1(x,y) &:= E(x,y) \\
P_2(x,y) &:= \exists z_1(E(x,z_1) \wedge P_1(z_1,y)) \\
P_3(x,y) &:= \exists z_2(E(x,z_2) \wedge P_2(z_2,y)) \\
&\vdots \\
P_{i+1}(x,y) &:= \exists z_i(E(x,z_i) \wedge P_i(z_i,y)).
\end{aligned}
$$

and then

$$\forall x(\neg P_3(x,x) \wedge \neg P_5(x,x) \wedge \cdots \wedge \neg P_{2\lceil n/2\rceil -1}(x,x)).$$

Counting logic with k variables:

C^k: collection of formulas for which
all subformulas have at most k free variables.

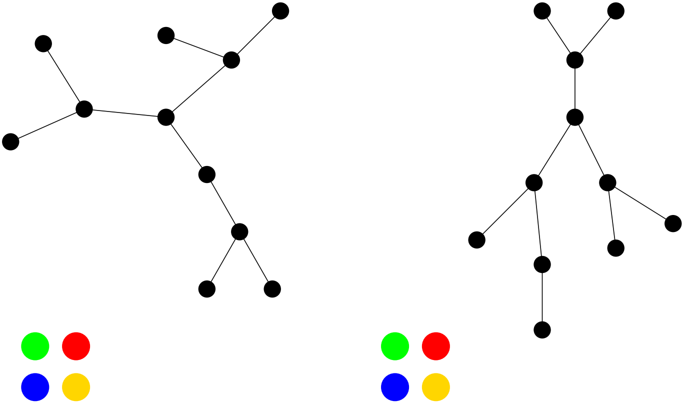
Indistinguishability by C^k

Definition:

$$G \cong_{\mathrm{C}}^{k} H \text{ iff for every } \phi \in C^k \text{ we have } G \models \phi \Leftrightarrow H \models \phi.$$

Pebble game (without counting moves)

Forced win for Spoiler.



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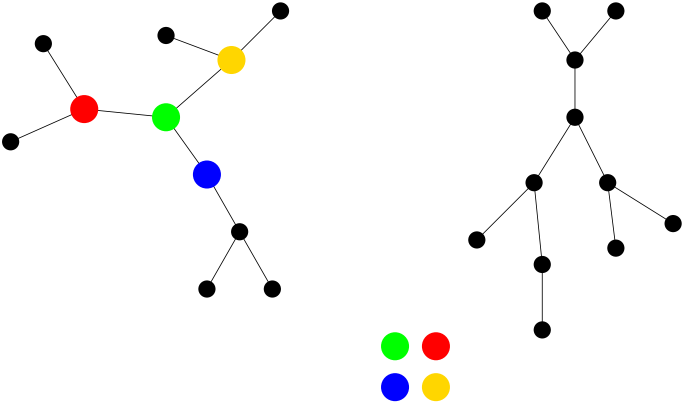

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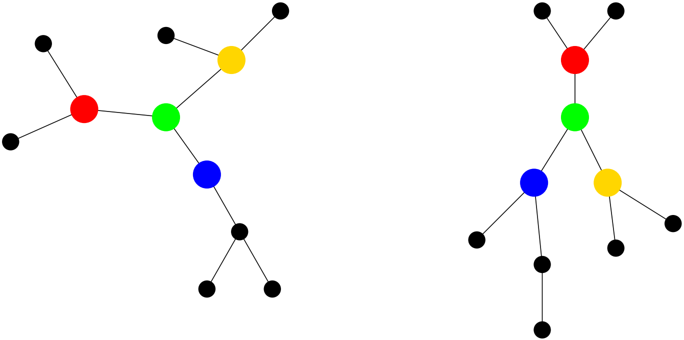

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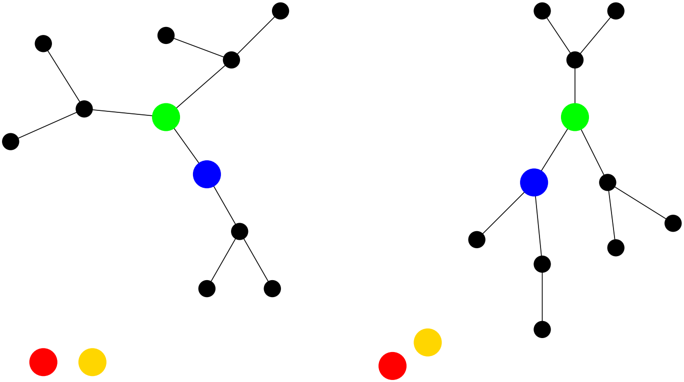

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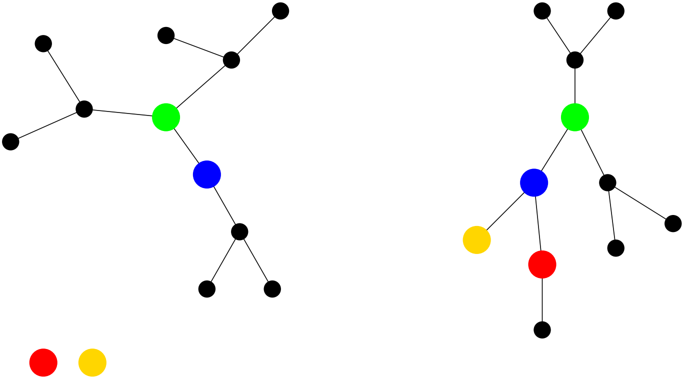

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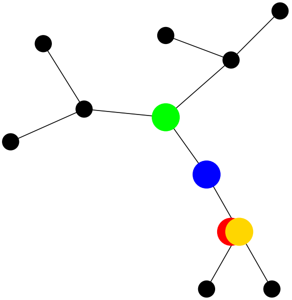
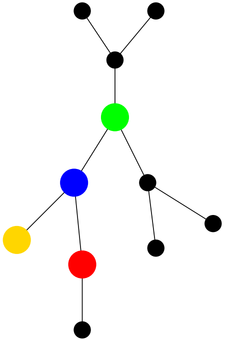
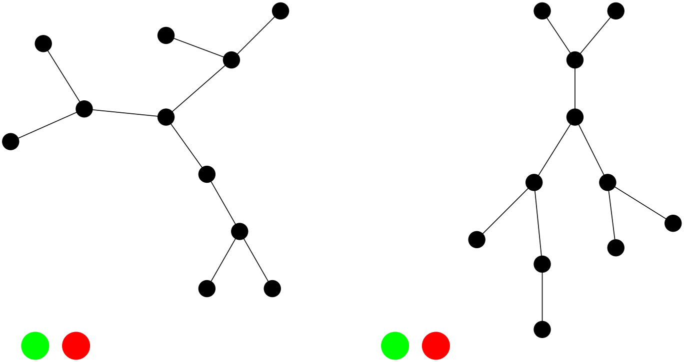




Pebble game WITH counting moves

Forced win for Spoiler.



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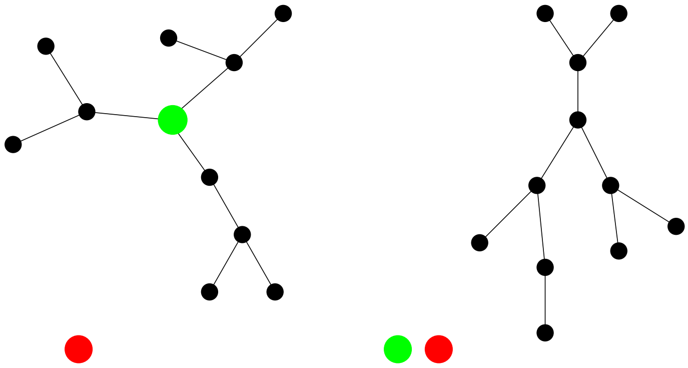

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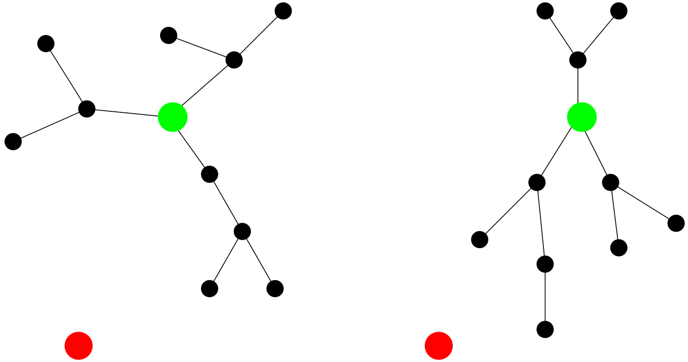

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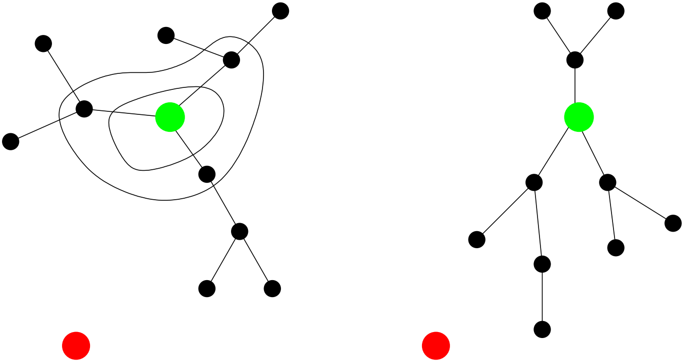

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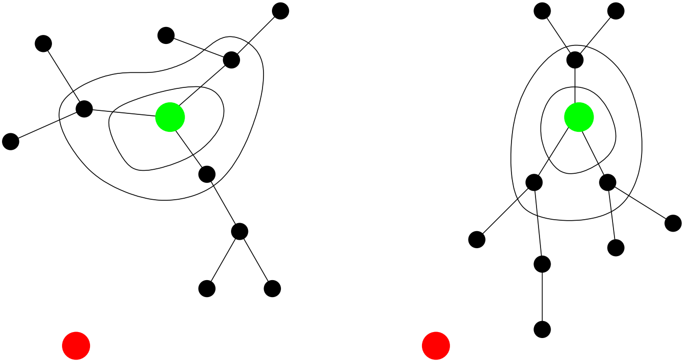

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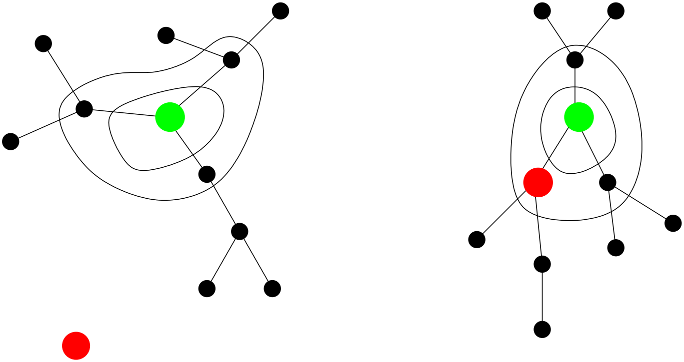

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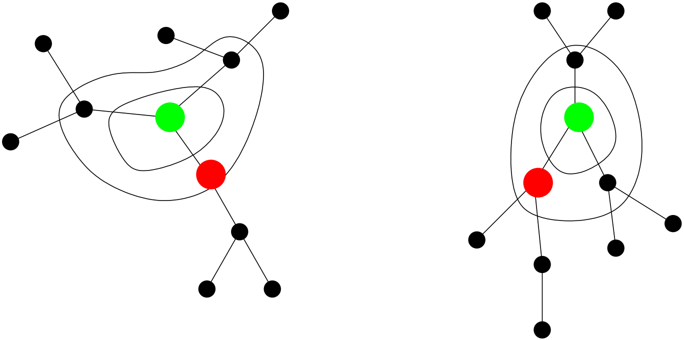

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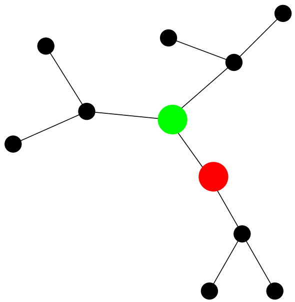

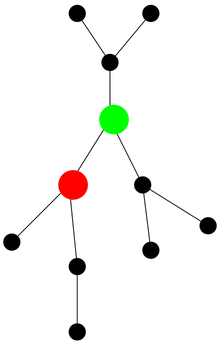

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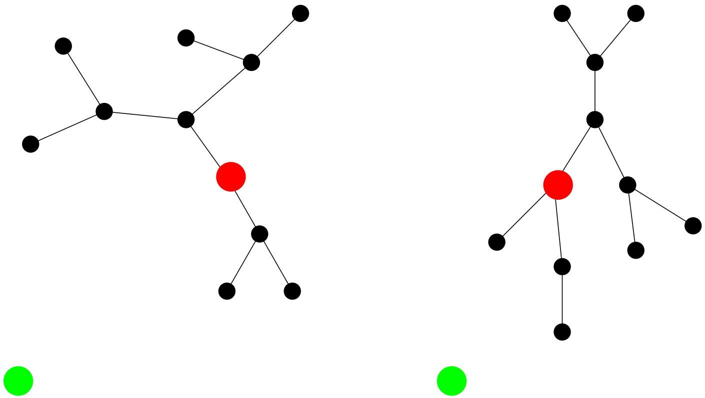

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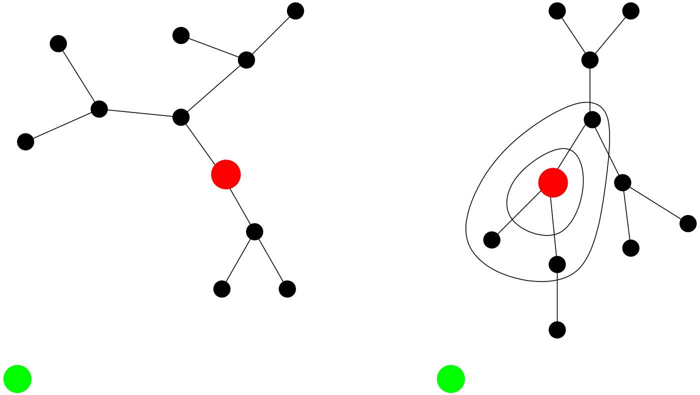

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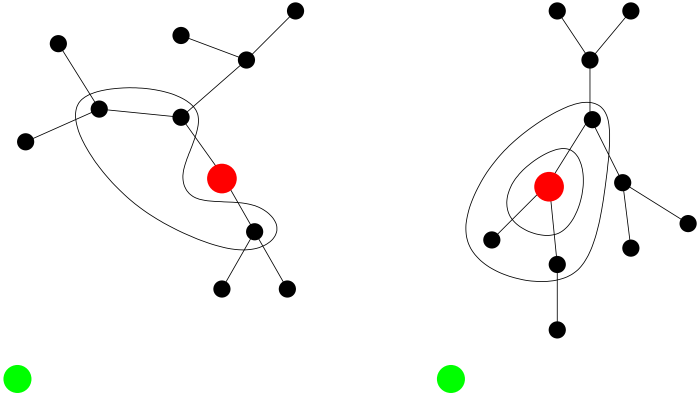

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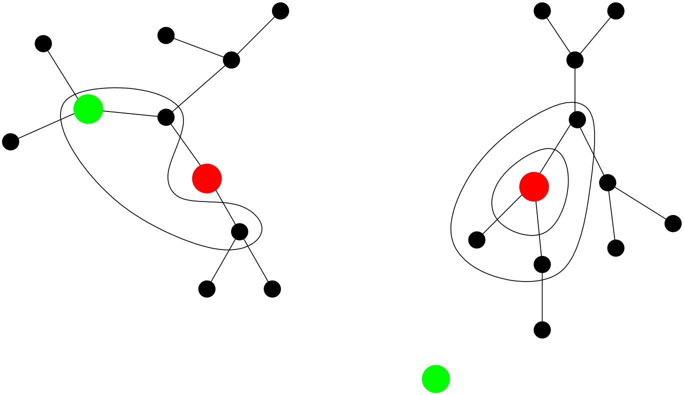

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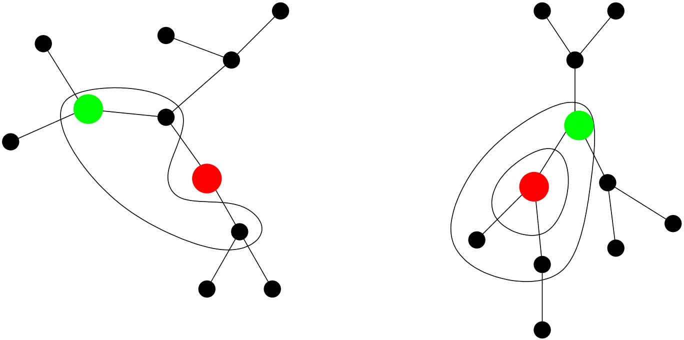

Systems with the back-and-forth properties

A winning strategy for the Duplicator in $G \cong_{\mathrm{C}}^{k} H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from G to H such that for every $f \in \mathcal{F}$ we have:

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1. (bounded) $|\mathrm{Dom}(f)| \leq k$,
2. (subfunction) For every $g \subseteq f$ we have $g \in \mathcal{F}$,
3. (back) If $|\mathrm{Dom}(f)| < k$ then:

   for every $X \subseteq V_G$ there exists $Y \subseteq V_H$ with $|Y| = |X|$ s.t.
   for every $v \in Y$ there exists $u \in X$ with $f \cup \{(u, v)\} \in \mathcal{F}$,

Systems with the back-and-forth properties

A winning strategy for the Duplicator in $G \cong_{\mathrm{C}}^{k} H$ is a non-empty collection $\mathcal{F}$ of partial isomorphisms from G to H such that for every $f \in \mathcal{F}$ we have:

1. (bounded) $|\mathrm{Dom}(f)| \leq k$,
2. (subfunction) For every $g \subseteq f$ we have $g \in \mathcal{F}$,
3. (back) If $|\mathrm{Dom}(f)| < k$ then:

   for every $X \subseteq V_G$ there exists $Y \subseteq V_H$ with $|Y| = |X|$ s.t.
   for every $v \in Y$ there exists $u \in X$ with $f \cup \{(u, v)\} \in \mathcal{F}$,
4. (forth) If $|\mathrm{Dom}(f)| < k$ then:

   for every $Y \subseteq V_H$ there exists $X \subseteq V_G$ with $|X| = |Y|$ s.t.
   for every $u \in X$ there exists $v \in Y$ with $f \cup \{(u, v)\} \in \mathcal{F}$.

Counting pebble game vs. Weisfeiler-Lehman algorithm

Theorem [Immerman-Lander 90, Cai-Fürer-Immerman 92]

$$G \cong^k_{\mathrm{WL}} H \Longleftrightarrow G \cong^{k+1}_{\mathrm{C}} H.$$

Relevant note: From its definition, it is not even obvious that $G \cong^k_{\mathrm{C}} H$ is decidable in time $n^{O(k)}$.

From feasible solutions to systems with B&F

Wanted:

$$G \cong^k_{\mathrm{SA}} H \Longrightarrow G \cong^k_{\mathrm{C}} H$$

Ingredient 1:

Birkhoff decomposition theorem: every doubly stochastic matrix is a convex combination of permutation matrices.

Ingredient 2:

Permutations preserve sizes of sets.

From systems with B&F to feasible solutions

Wanted:

$$G \cong^k_{\mathrm{C}} H \Longrightarrow G \cong^{k-1}_{\mathrm{SA}} H$$

Ingredient 1:

A sliding game to account for $AS = SB$;
here is where the -1 is lost.

Ingredient 2:

Normalizing winning strategies into uniform ones.

Part V

APPLICATIONS (or what to do of this?)

Isomorphism testing for special graphs

Theorem [Immerman-Lander 90, Grohe 98, ...]

1. If G is a tree, then $G \cong^2_{\mathrm{C}} H$ iff $G \cong H$, for every H.
2. If G is planar, then $G \cong^{15}_{\mathrm{C}} H$ iff $G \cong H$, for every H.
3. $\cdots$

Corollary

For all such graph classes,
an explicit and poly-size LP
solves graph isomorphism.

SA-rank lower bounds

Consider the standard LP-relaxation of vertex cover:

$$
\begin{aligned}
&\text{minimize } \textstyle\sum_{u \in V} x_u \\
&\text{subject to} \\
&\quad x_u + x_v \geq 1 \qquad \text{for every } (u, v) \in E, \\
&\quad x_u \geq 0 \qquad\qquad \text{for every } u \in V.
\end{aligned}
$$

SA-rank lower bounds

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minimize $\sum_{u \in V} x_u$
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We expect that the inequality

$$\sum_{u \in V} x_u \geq \mathrm{vc}(G) \tag{1}$$

will not, in general, be valid over $P^k(G)$ for any $k = O(1)$.

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will not, in general, be valid over $P^k(G)$ for any $k = O(1)$.
Indeed:

Theorem [exercise, also follows Schoenebeck 08]

There exist graphs G for which (2) is not valid over $P^{\Omega(n)}(G)$.

New proof

Sketch:

1. Start with the n-vertex CFI graphs $G \cong_{\mathrm{C}}^{\Omega(n)} H$ yet $G \ncong H$.
2. In particular $(G, G) \cong_{\mathrm{C}}^{\Omega(n)} (G, H)$ yet $G \cong G$ and $G \ncong H$.
3. Apply the reduction from graph isomorphism to vertex cover.
4. Get graphs $A \cong_{\mathrm{C}}^{\Omega(n)} B$ with $\mathrm{vc}(A) \neq \mathrm{vc}(B)$.
5. Apply transfer lemma and get $A \cong_{\mathrm{SA}}^{\Omega(n)} B$.

Final step:

$$A \cong_{\mathrm{SA}}^{2k} B \Longrightarrow \mathrm{opt}(P^k(A)) = \mathrm{opt}(P^k(B)).$$

SA-rank lower bounds

Consider the standard LP-relaxation of max-cut:

$$\text{maximize } \tfrac{1}{2} \textstyle\sum_{uv \in E} x_{uv}$$

subject to

$$x_{uv} = x_{vu}$$

$$x_{uw} \leq x_{uv} + x_{vw}$$

$$x_{uv} + x_{vw} + x_{wu} \leq 2$$

$$0 \leq x_{uv} \leq 1$$

SA-rank lower bounds

Consider the standard LP-relaxation of max-cut:

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&\quad x_{uw} \leq x_{uv} + x_{vw} \\
&\quad x_{uv} + x_{vw} + x_{wu} \leq 2 \\
&\quad 0 \leq x_{uv} \leq 1
\end{aligned}
$$

We expect that the inequality

$$\sum_{u \in V} x_u \leq \mathrm{mc}(G) \qquad (2)$$

will not, in general, be valid over $P^k(G)$ for any $k = O(1)$.

SA-rank lower bounds

Consider the standard LP-relaxation of max-cut:

$$\text{maximize } \tfrac{1}{2}\sum_{uv\in E} x_{uv}$$

subject to

$$x_{uv} = x_{vu}$$

$$x_{uw} \le x_{uv} + x_{vw}$$

$$x_{uv} + x_{vw} + x_{wu} \le 2$$

$$0 \le x_{uv} \le 1$$

We expect that the inequality

$$\sum_{u\in V} x_u \le \mathrm{mc}(G) \qquad (2)$$

will not, in general, be valid over $P^k(G)$ for any $k = O(1)$.
Indeed:

Theorem [follows from Schoenebeck 08]

There exist graphs G for which (2) is not valid over $P^{\Omega(n)}(G)$.

New proof

Sketch:

1. Start with the n-vertex CFI graphs $G \cong_{\mathrm{C}}^{\Omega(n)} H$ yet $G \not\cong H$.
2. In particular $(G, G) \cong_{\mathrm{C}}^{\Omega(n)} (G, H)$ yet $G \cong G$ and $G \not\cong H$.
3. Apply the reduction from graph isomorphism to max-cut.
4. Get graphs $A \cong_{\mathrm{C}}^{\Omega(n)} B$ with $\mathrm{mc}(A) \neq \mathrm{mc}(B)$.
5. Apply transfer lemma and get $A \cong_{\mathrm{SA}}^{\Omega(n)} B$.

Final step:

$$A \cong_{\mathrm{SA}}^{3k} B \Longrightarrow \mathrm{opt}(P^k(A)) = \mathrm{opt}(P^k(B)).$$

Local LPs

Basic k-local LPs:

1. one variable $x_{\mathbf{u}}$ for each k-tuple $\mathbf{u} \in V^k$,
2. one inequality $\sum_{\mathbf{u} \in V^k} a_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}} \geq b_{\mathbf{v}}$ for every k-tuple $\mathbf{v} \in V^k$,
3. coefficients $a_{\mathbf{u},\mathbf{v}}$ depend only on the type $\mathrm{tp}_G(\mathbf{u}, \mathbf{v})$,
4. coefficients $b_{\mathbf{v}}$ depend only on the type $\mathrm{tp}_G(\mathbf{v})$.

Generic k-local LPs:

Unions of generic basic k-local LPs
(with coefficients given as a function of the types).

Instantiation of generic k-local LPs:

Let P is a generic k-local LP.
Then $P(G)$ is the LP associated to G.

Metric polytope

Recall the metric polytope:

$$\frac{1}{2}\sum_{uv\in E} x_{uv} \geq W$$
$$x_{uv} = x_{vu}$$
$$x_{uw} \leq x_{uv} + x_{vw}$$
$$x_{uv} + x_{vw} + x_{uw} \leq 2$$
$$0 \leq x_{uv} \leq 1$$

1. Objective function: basic 2-local LP
2. Symmetry constraint: two basic 2-local LPs
3. Triangle inequality: basic 3-local LP
4. Perimetric inequality: basic 3-local LP
5. Unit cube constraint: two basic 2-local LPs

Preservation of local LPs

Theorem: Let P be a generic k-local LP.

If $G \cong^k_{\mathrm{SA}} H$, then $P(G)$ is feasible iff $P(H)$ is feasible.

'Just do it' proof:

1. Let $\{x_{\mathbf{u}}\}$ be a feasible solution for $P(G)$.
2. Let $\{X_{\mathbf{u},\mathbf{v}}\}$ be a feasible solution for $I(G,H)^k$.
3. Define:
$$y_{\mathbf{v}} := \sum_{\mathbf{u} \in G^k} X_{\mathbf{u},\mathbf{v}} \cdot x_{\mathbf{u}}.$$
4. Check that $\{y_{\mathbf{v}}\}$ is a feasible solution for $P(H)$.

More examples of local LPs

More examples:

1. maximum flows (2-local)
2. matchings on bipartite graphs (2-local)
3. relaxation of max-cut via the metric polytope (3-local)
4. relaxation of vertex cover (2-local)
5. r SA-levels of k-local LPs are $O(kr)$-local LPs.

Expressibility results

Consider the max-flow LP. It is 2-local. It is integral.

Corollary

$$G \cong^3_{\mathrm{C}} H \Rightarrow \mathrm{mf}(G) = \mathrm{mf}(H).$$

Corollary

There exists a sentence in C^3 that,
over *st*-networks with n vertices, defines those
whose maximum flow is at least the out-degree of the source.

Expressibility results

Consider the metric polytope again.

Theorem [Barahona-Majoub 86]:

If G is a K_5 minor-free graph, then $\mathrm{mc}(G) = \mathrm{opt}(P(G))$.

Corollary

If G and H are K_5 minor-free, then $G \cong^4_{\mathrm{C}} H \Rightarrow \mathrm{mc}(G) = \mathrm{mc}(H)$.

Corollary

There exists a sentence in C^4 that,
over K_5 minor-free n-vertex graphs, defines those
whose max-cut is at least $n/4$.

Part VI

DISCUSSION AND OPEN PROBLEMS

Get new rank lower bounds from inexpressibility results?

Challenging problem:

Prove that an integrality gap of $2 - \epsilon$ resists $\Omega(n)$ SA-levels of vertex-cover.

Get new rank lower bounds from inexpressibility results?

Challenging problem:

Prove that an integrality gap of $2 - \epsilon$ resists $\Omega(n)$ SA-levels of vertex-cover.

What would be enough?:

Find G and H such that:

1. $\mathrm{mc}(G) \geq (2 - \epsilon) \cdot \mathrm{mc}(H)$
2. $G \cong_{\mathrm{C}}^{\Omega(n)} H$.

New expressibility/inexpressibility results?

Challenging problem:

Is perfect matching definable in $C^{O(1)}$?
(answer is YES for bipartite graphs)

New expressibility/inexpressibility results?

Challenging problem:

Is perfect matching definable in $C^{O(1)}$?
(answer is YES for bipartite graphs)

SOLVED! [Anderson-Dawar-Holm 13]:

YES even for general graphs!

TODA!