GAPS BETWEEN
CLASSICAL SATISFIABILITY PROBLEMS
AND
THEIR QUANTUM RELAXATIONS

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Constraint Satisfaction Problems (CSPs)

Variables and Values:

$$V = \{X_1 \ldots, X_n\} \text{ and } D = \{b_1, \ldots, b_q\}$$

System of constraints:

$$R_1(t_1), \ldots, R_m(t_m)$$

where

$R_j \subseteq D^{r_j}$ is the constraint relation

$t_j \in V^{r_j}$ is the constraint scope

Solution space:

$$f : V \to D \text{ with } f(t_j) \in R_j \text{ for every } j = 1, \ldots, m$$

Example 1:

System of linear equations over $\mathbb{Z}_2$:

$$\begin{aligned} X_1 + X_2 + X_3 &\equiv 0 \pmod 2 \\ X_2 + X_4 + X_5 &\equiv 1 \pmod 2 \\ X_3 + X_4 + X_2 &\equiv 1 \pmod 2 \end{aligned}$$

Here

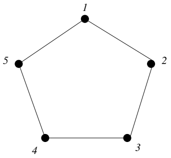
$$V = \{X_1, X_2, X_3, X_4, X_5\} \text{ and } D = \mathbb{Z}_2,$$

and the constraint relations are

$$\begin{aligned} R_0 &= \{(a, b, c) \in D^3 : a + b + c \equiv 0 \pmod 2\} \\ R_1 &= \{(a, b, c) \in D^3 : a + b + c \equiv 1 \pmod 2\} \end{aligned}$$

Example 2

Graph 3-colorability:



$$\begin{array}{l} X_1 \neq X_2 \\ X_2 \neq X_3 \\ X_3 \neq X_4 \\ X_4 \neq X_5 \\ X_5 \neq X_1 \end{array} \quad \text{with } X_i \in \{{\color{red}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet}\}$$

Here

$$V = \{X_1, X_2, X_3, X_4, X_5\},\ D = \{{\color{red}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet}\},\ \text{and } R = \text{``} \neq \text{''}.$$

Questions Concerning CSPs

1. Satisfiability: Does it have a solution? (k-SAT, k-colorability, systems of equations)
2. Optimization: How many constraints can be satisfied simultaneously? (MAX-3-SAT, MAX-CUT, unique games)
3. Counting: How many solutions does it have? ($\#$-SAT, computing partition functions of spin systems)
4. Structure: Is the space of solutions connected through single value-flips? (sampling by Monte-Carlo Markov chain)
5. Relaxations: When is a certain relaxation of the problem exact? (LP relaxation, SDP relaxation, ...)
6. ...

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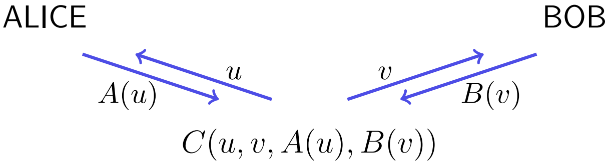
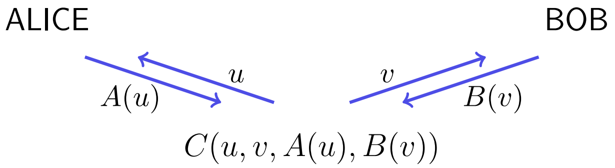
This talk:

QUANTUM RELAXATIONS

Non-local Games



Non-local Games

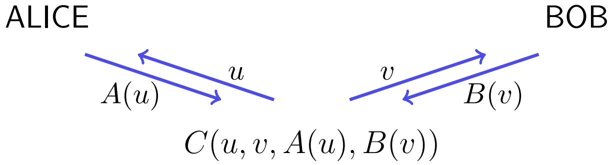


Game:

π : probability distribution on $U \times V$

$C : U \times V \times R \times S \to \{0, 1\}$

Non-local Games



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Strategies:

$$A : U \to R$$

$$B : V \to S$$

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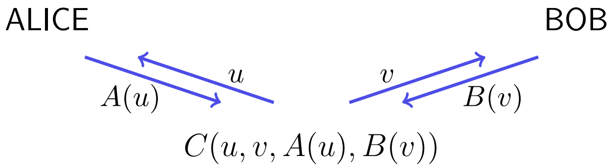

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Value of the game:

$$\max_{A,B} \mathop{\mathbb{E}}_{(u,v)} \left[C(u, v, A(u), B(v)) \right]$$

CSPs as Non-local Games

$$R_1(t_1), \ldots, R_m(t_m)$$

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Verifier randomly chooses $j \in \{1, \ldots, m\}$ and sends it to Alice.
Verifier randomly chooses i with X_i in t_j and sends it to Bob.

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Alice replies with an assignment of values to t_j satisfying R_j.
Bob replies with an assignment of value to X_i.

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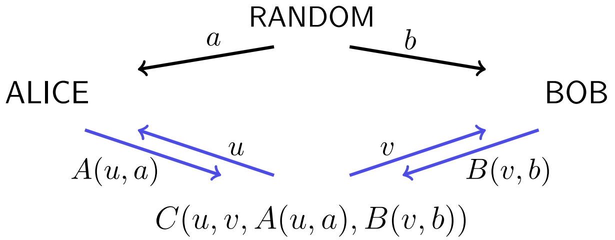
Fact: The following are equivalent:

1. The instance is satisfiable.
2. Value of the game is 1.

Non-local Games with Randomness



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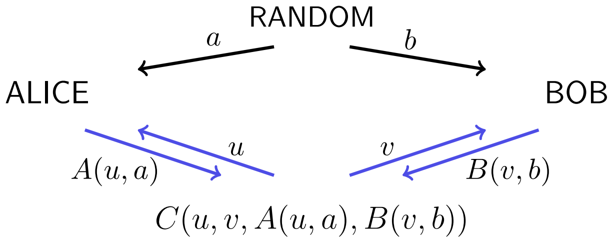

Strategies:

σ : probability distribution on $W_A \times W_B$

$A : U \times W_A \to R$

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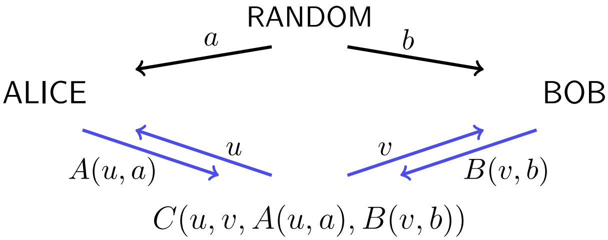

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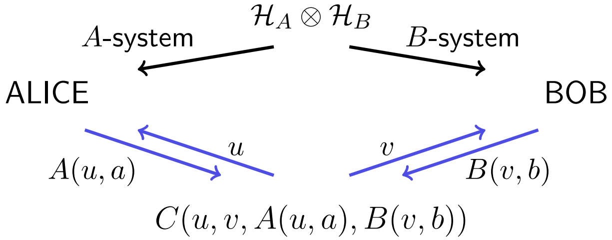
Value of the game:

$$\max_{\sigma, A, B} \mathop{\mathbb{E}}_{(u,v)} \mathop{\mathbb{E}}_{(a,b)} \left[V(u, v, A(u, a), B(v, b)) \right]$$

Non-local Games with Entanglement



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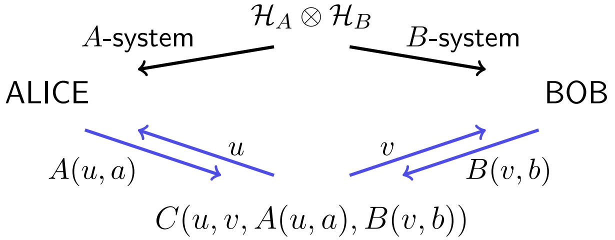


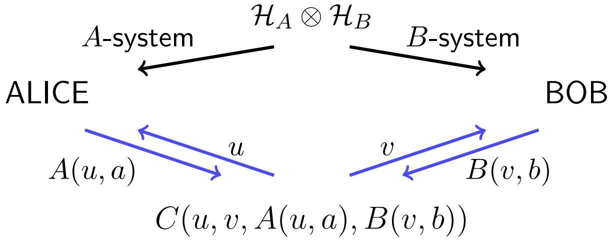
Strategies:

Φ : unit vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ (a quantum state)

$A : U \times O_A \to R$ based on measuring A-system

$B : V \times O_B \to S$ based on measuring B-system

Non-local Games with Entanglement



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Bell's Theorem

Fact:

$$\text{Deterministic value} \leq \text{Randomized value} \leq \text{Quantum value}$$

Theorem [Bell 1964]
There exists a game such that

$$\frac{\text{Randomized value}}{\text{Quantum value}} = 0.87856...$$

Mermin's Theorem: Our Starting Point

Theorem [Mermin 1993]
There exists a system of linear equations over $\mathbb{Z}_2$ such that, for the corresponding non-local game:

$$\text{Randomized value} < 1$$
$$\text{Quantum value} = 1.$$

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There exists a system of linear equations over $\mathbb{Z}_2$ such that, for the corresponding non-local game:

Randomized value < 1

Quantum value $= 1$.

$$\begin{array}{ccccc} X_{11} & X_{12} & X_{13} & = & +1 \\ X_{21} & X_{22} & X_{23} & = & +1 \\ X_{31} & X_{32} & X_{33} & = & -1 \\ \| & \| & \| & & \\ +1 & +1 & +1 & & \end{array}$$

Boolean Constraint Languages

Boolean domain: $\{\pm 1\}$ with $+1 =$ *false* and and $-1 =$ *true*;
Constraint language: a set A of relations $R \subseteq \{\pm 1\}^r$

relations $\leftrightarrow$ predicates $\leftrightarrow$ polynomials

characteristic function

Fourier-Welsh transform

Examples:

OR	disjunctions of literals
LIN	linear equations over $\mathbb{Z}_2$
1-IN-3	triples with one -1 and two $+1$ components
NAE	triples with not-all-equal components

Generalized Satisfiability Problems: SAT(A)

$$\exists X_1 \cdots X_n (C_1 \wedge \cdots \wedge C_m)$$

variables $X_1, \ldots, X_n$ range over $\{\pm 1\}$

constraints $C_1, \ldots, C_m$ each of the form $R(Y_1, \ldots, Y_r) = 1$

in A

X_i's or ± 1

Examples:

3-SAT	1-IN-3-SAT
HORN-SAT	NAE-SAT
LIN-SAT	...

[Schaefer 1978]

... via Operator Assignments

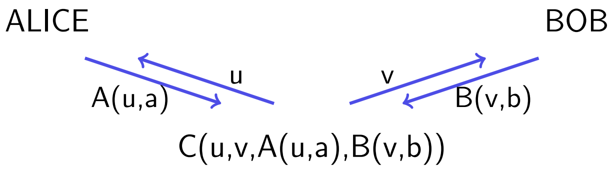
$$\exists X_1 \cdots X_n (C_1 \wedge \cdots \wedge C_m)$$

variables $X_1, \ldots, X_n$
range over $B(\mathcal{H})$,
the linear operators
of a Hilbert space $\mathcal{H}$

constraints $C_1, \ldots, C_m$ each
of the form $R(Y_1, \ldots, Y_r) = I$
$Y_i Y_j = Y_j Y_i$ for all $i, j \in [r]$
and
$X_i^2 = I$ for all $i \in [n]$
(multiplication = composition)

SAT(A)	satisfiability over $\mathbb{C}$ (i.e., over $\{\pm 1\}$ by •)
SAT*(A)	satisfiability over some finite-dimensional $\mathcal{H}$
SAT**(A)	satisfiability over some arbitrary $\mathcal{H}$

Back to Games with Entangled Players



Theorem [Cleve-Mittal 2014, Cleve-Liu-Slofstra 2016]

SAT $\leftrightarrow$ classical strategies
SAT* $\leftrightarrow$ quantum strategies in tensor product model
SAT** $\leftrightarrow$ quantum strategies in commuting operator model

Gap Instances

Mermin-Peres Magic Square

$$
\begin{array}{llll}
X_{11}\,X_{12}\,X_{13} & = +1 \\
X_{21}\,X_{22}\,X_{23} & = +1 \\
X_{31}\,X_{32}\,X_{33} & = -1 \\
\parallel \quad \parallel \quad \parallel & \\
+1 \quad +1 \quad +1 &
\end{array}
$$

Unsatisfiable SAT-instance of LIN

Satisfiable SAT*-instance of LIN

↓

a SAT-vs-SAT* gap for LIN

SAT-vs-SAT*	gap of the first kind
SAT-vs-SAT**	gap of the second kind
SAT*-vs-SAT**	gap of the third kind

Gaps of first kind for LIN exist [Mermin 1990]

Gaps of third kind for LIN exist [Slofstra 2017]

Gaps of first kind for 2-SAT or HORN do not exist [Ji 2014]

Classification

Theorem [A.-Kolaitis-Severini 2017]
For every Boolean constraint language A,

1. either gaps of every kind for A exist,
2. or gaps of no kind for A exist.

Moreover:

gaps for A do not exist
iff
A is of one of the following types:
iff
LIN is **not** pp-definable from A

- 0-valid
- 1-valid
- Horn
- dual Horn
- bijunctive

Primitive Positive Definitions

$$R(Y_1, \ldots, Y_r) \quad \equiv \quad \exists Z_1 \cdots \exists Z_s (C_1 \wedge \cdots \wedge C_t)$$

auxiliary variables

constraints on the Y's and Z's

Example:

$$\mathsf{NAE}(X, Y, Z) \;\equiv\; (X \vee Y \vee Z) \wedge (\overline{X} \vee \overline{Y} \vee \overline{Z})$$

Proof Recipe

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Ingredient 1: gap preserving reductions

Lemma:
If A is pp-definable from B,
then gaps for B imply gaps for A.

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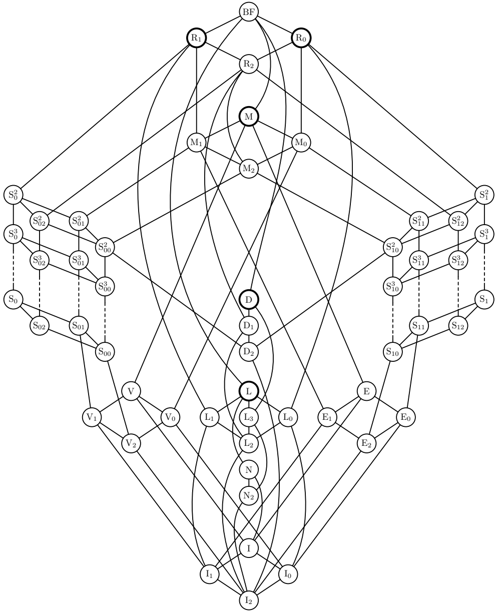
Lemma:
If A is pp-definable from B,
then gaps for B imply gaps for A.

Ingredient 2: Post's Lattice of Boolean co-clones

Theorem [Post 1941]:
There are countably many Boolean constraint languages up to pp-definability, and we know them.

Post's Lattice



More on Primitive Positive Definability

$$R(Y_1, \ldots, Y_r) \quad \equiv \quad \exists Z_1 \cdots \exists Z_s (C_1 \wedge \cdots \wedge C_t)$$

pp-def Z_i's range over $B(\mathbb{C})$ (i.e., over $\{\pm 1\}$ by $Z_i^2 = I$)
pp*-def Z_i's range over $B(\mathcal{H})$, for some finite-dim $\mathcal{H}$
pp**-def Z_i's range over $B(\mathcal{H})$, for some arbitrary $\mathcal{H}$

A Conservativity Theorem

Theorem [A.-Kolaitis-Severini 2017]:
For every two constraint languages A and B,
the following statements are equivalent.

1. every relation in A is pp-definable from B
2. every relation in A is pp*-definable from B

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Corollary: OR is **not** pp*-definable from LIN

Closure Operations via Operators

R is invariant under $F : \mathcal{H}_1 \times \cdots \times \mathcal{H}_r \to \mathcal{H}$ if

$$\frac{\begin{array}{l} R(\;A_{1,1}\;,\cdots,\;A_{1,r}\;) = I \text{ and commute} \\ \qquad\;\vdots \qquad\quad \ddots \qquad \vdots \\ R(\;A_{s,1}\;,\cdots,\;A_{s,r}\;) = I \text{ and commute} \end{array}}{R(F(\mathbf{A}_{*,1}),\cdots,F(\mathbf{A}_{*,r})) = I \text{ and commute}}$$

Lemma: If A is invariant under $F : \{\pm 1\}^s \to \{\pm 1\}$, then every $R \subseteq \{\pm 1\}^r$ pp*-definable from A is invariant under

$$F^*(X_1, \ldots, X_s) := \sum_{S \subseteq [s]} \widehat{F}(S) \bigotimes_{i=1}^{s} X_i^{S(i)}$$

Proof by Example

$$
\begin{array}{cccl}
X_{11} & X_{12} & X_{13} & = +1 \\
X_{21} & X_{22} & X_{23} & = +1 \\
X_{31} & X_{32} & X_{33} & = +1 \\
\rotatebox{90}{$=$} & \rotatebox{90}{$=$} & \rotatebox{90}{$=$} & \\
\rotatebox{-90}{$+1$} & \rotatebox{-90}{$+1$} & \rotatebox{-90}{-1} &
\end{array}
$$

Proof by Example

$$\begin{array}{cccl} X_{11} & X_{12} & X_{13} & = +1 \\ X_{21} & X_{22} & X_{23} & = +1 \\ X_{31} & X_{32} & X_{33} & = +1 \\ \| & \| & \| & \\ +1 & +1 & -1 & \end{array}$$

$$(X_{11} \otimes X_{21} \otimes X_{31})(X_{12} \otimes X_{22} \otimes X_{32})(X_{13} \otimes X_{23} \otimes X_{33}) =$$
$$(X_{11}X_{12}X_{13}) \otimes (X_{21}X_{22}X_{23}) \otimes (X_{31}X_{32}X_{33}) =$$
$$(+1) \otimes (+1) \otimes (+1) =$$
$$+1$$

Future Work

Question 1:

Classification of gaps for q-valued domains with $q > 2$?

Question 2:

Is SAT*(LIN) decidable?
(Note: Slofstra proved that SAT**(LIN) is undecidable)

Question 3:

Closure operators, fine. Identities?

Question 4:

Is pp**-definability = pp-definability also?

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