ON THE EXISTENCE OF NORMAL FORMS FOR LOGICS THAT CAPTURE COMPLEXITY CLASSES

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We show that the logic $\mathrm{HEX}^*[\mathrm{FO}_s]$, which is first-order logic (with successor) extended with a generalized quantifier (HEX) corresponding to the **PSPACE**–complete problem Generalized Hex, has a projective normal form. This gives as a corollary that $\mathrm{HEX}^*[\mathrm{FO}_s]$ captures **PSPACE** and that the problem Generalized Hex is complete via first-order projections. We define a variation of HEX, namely WHEX, and prove that as a problem (class of finite structures) it is also complete for **PSPACE** via logspace reducibility. Considering WHEX as a generalized quantifier, we form the logic $\mathrm{WHEX}^*[\mathrm{FO}_s]$ and show that it has similar properties as $\mathrm{HEX}^*[\mathrm{FO}_s]$; that is, it has a projective normal form and captures **PSPACE**; hence, WHEX is complete for **PSPACE** via first-order projections. Furthermore, we show that $\mathrm{WHEX}^*[\mathrm{FO}_s]$ is contained in the infinitary logic $\mathrm{L}^{\omega}_{\infty\omega}$.

We then study the existence of normal forms for the logics $\mathrm{HEX}^*[\mathrm{FO}]$ and $\mathrm{WHEX}^*[\mathrm{FO}]$, where the successor relation is not present, and conclude that they have no projective normal forms. In fact, we prove a general theorem that states the non-existence of a projective normal form for various extensions of first-order logic (without successor) of the form $\Omega^*[\mathrm{FO}]$, when Ω is a generalized quantifier that corresponds to a firmly monotone problem. The tool we employ for obtaining our negative results is Ehrenfeucht-Fraïssé games for logics with generalized quantifiers.

We continue our explorations in the world of unordered structures with a revision of a theorem by Grädel, which states the existence of a hierarchy of logics inside Transitive Closure logic, or $\mathrm{TC}^*[\mathrm{FO}]$. We translate this result

to the framework of program schemes, and obtain a hierarchy of certain classes of program schemes. Our Hierarchy Theorem for program schemes generalizes Grädel's Hierarchy Theorem for $\mathrm{TC}^*[\mathrm{FO}]$, since, as we show, his theorem can be obtained as a corollary from ours, and similar hierarchies inside other logics of the form $\Omega^*[\mathrm{FO}]$, where Ω is a generalized quantifier of complexity within **NL**, are also consequences of our theorem.