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## Generalized Hex and logical characterizations of polynomial space

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## Generalized Hex and logical characterizations of polynomial space

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### Abstract

We extend a logical characterization of PSPACE due to Makowsky and Pnueli by showing that their logic has a particular normal form which implies that the Generalized Hex problem is complete for PSPACE via very restricted logical reductions. We also show that this normal form result fails in the absence of a built-in successor relation. © 1997 Elsevier Science B.V.

**Keywords:** Computational complexity; Descriptive complexity; Finite model theory; Logical characterizations of polynomial space; Completeness via logical reductions

We consider a particular logical characterization of the complexity class PSPACE using first-order logic, with a built-in successor relation, extended with an operator corresponding to the well-known PSPACE-complete decision problem Generalized Hex; that is, the logic  $(\pm\text{HEX})^*[\text{FO}_s]$ . It was shown by Makowsky and Pnueli [12] (see also [11]) that any problem in PSPACE can be defined by a sentence of the logic  $(\pm\text{HEX})^*[\text{FO}_s]$ , and, conversely, that any problem definable by a sentence of this logic is in PSPACE.

There are numerous other similar logical characterizations of complexity classes (that is, using logics obtained by extending first-order logic, with suc-

cessor, using operators, or, more precisely, uniform sequences of Lindström quantifiers, corresponding to problems), the first such being Immerman's characterization of NL as those problems definable in (the now well-studied) transitive closure logic,  $(\pm\text{TC})^*[\text{FO}_s]$ . To our knowledge, for all of these other characterizations, more information is forthcoming; that is, the logics involved possess normal forms and these normal forms yield strong complexity-theoretic completeness results. However, Makowsky and Pnueli's logical characterization of PSPACE failed to establish such a normal form for the logic  $(\pm\text{HEX})^*[\text{FO}_s]$  and they left it as an open problem as to whether the normal form existed. In Theorem 1 of this note, we establish such a normal form for  $(\pm\text{HEX})^*[\text{FO}_s]$  which yields as an immediate corollary that HEX is complete for PSPACE via quantifier-free projections (also called projection translations) with successor.

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<sup>1</sup> Most of this work was done whilst the author was visiting the University of Leicester.

Other problems have been shown to be complete for PSPACE via quantifier-free translations with successor in [2,8,9]. However, these problems are rather “unnatural”, being based around the logical characterization of PSPACE as partial fixed point logic with successor [1], in the sense that first-order logic was augmented with a contrived operator to try and mimic the application of the partial-fixed point construct. On the other hand, the normal form results for the logics in [2,8,9] hold in the absence of a successor relation (which was the whole point of the research in those papers). A complete problem for PSPACE via quantifier-free projections with successor was also exhibited in [15] (although it was not explicitly stated there as being so) but again this problem was “unnatural”, being based around a characterization of PSPACE using a different inductive construct. Our result that HEX is complete for PSPACE via quantifier-free projections (or first-order translations, for that matter) with successor is the first such completeness result involving what could be called a “natural” problem.

Notwithstanding the preceding paragraph, to our mind, our actual proof of Theorem 1 is the most interesting aspect of this note given that it is essentially identical to Immerman’s proof in [10] that transitive closure logic (or, as was proven there, the positive version,  $TC^*[FO_s]$ ) has a normal form, except that in the combinatorial construction we replace an edge in one of Immerman’s digraphs with a particular “gadget” (see the proof of Theorem 1). This fact encourages one to view the problem HEX as a “game theoretic” counterpart to TC. We intend to investigate this phenomenon more closely in future and hope to obtain criteria under which one can “automatically” transform a normal form result for some logic  $(\pm\Omega)^*[FO_s]$  (which might capture NL, for example) to the logic formed using the “game-theoretic” version of  $\Omega$  (which might capture PSPACE, for example).

The complexity class PSPACE does have logical characterizations in which a successor relation, or any other built-in relation, does not appear (and consequently we have been very careful above in detailing when the successor relation is present in our logics). However, as yet no problem (natural or otherwise) has been shown to be complete for PSPACE via restricted logical reductions in the absence of the successor relation. Our final result in this note is that in

the absence of the successor relation, both the normal form and the logical characterization in Theorem 1 fail to hold. (Note that although the logics in [2,8,9] have normal form results, these logics do not capture PSPACE in the absence of a successor relation.)

Given that finite model theory and descriptive complexity theory are now firmly established in logic and theoretical computer science, rather than give definitions here we simply refer the reader: to the paper [14] for all definitions and concepts regarding logics of the form  $(\pm\Omega)^*[FO_s]$  and their relation to complexity classes; to the paper [17] for (generalized) Ehrenfeucht–Fraïssé games and their applicability to logics without built-in relations; and to the book [4] for background issues.

Now for our results. Let the signature  $\sigma_{2++} = \langle E, C, D \rangle$ , where  $E$  is a binary relation symbol and  $C$  and  $D$  are constant symbols (our signatures never contain function symbols). The problem HEX consists of those  $\sigma_{2++}$ -structures  $\mathcal{S}$  for which Player 1 has a winning strategy in the game of Generalized Hex on  $\mathcal{S}$ , where the game of Generalized Hex is played as follows. Starting with Player 1, two players take it in turns to colour previously uncoloured vertices of the graph described by  $E^{\mathcal{S}}$ , apart from  $C^{\mathcal{S}}$  and  $D^{\mathcal{S}}$ , with Player 1 using the colour blue and Player 2 using the colour red. If, at the end of the play, there is a path from the source,  $C^{\mathcal{S}}$ , to the sink,  $D^{\mathcal{S}}$ , consisting entirely of blue-coloured vertices then Player 1 wins; otherwise Player 2 wins. The notion of Player 1 having a winning strategy should be clear. The problem HEX is well known to be complete for PSPACE via logspace reductions (see [7] for more details).

The problem TC consists of those  $\sigma_{2++}$ -structures  $\mathcal{S}$  for which there is a path in the digraph described by  $E^{\mathcal{S}}$  from the source,  $C^{\mathcal{S}}$ , to the sink,  $D^{\mathcal{S}}$ .

**Theorem 1.**  $(\pm\text{HEX})^*[FO_s] = \text{PSPACE}$ , and every problem in PSPACE can be defined by a sentence of the form

$$\text{HEX}[\lambda x, y\psi(x, y)](\mathbf{0}, \text{max}),$$

where  $|x| = |y| = k$ , for some  $k$ ,  $\psi$  is a quantifier-free projection with successor, and  $\mathbf{0}$  respectively  $\text{max}$  is the constant symbol  $\mathbf{0}$  (respectively  $\text{max}$ ) repeated  $k$  times.

**Proof.** The result that  $(\pm\text{HEX})^*[\text{FO}_s] = \text{PSPACE}$  is due to Makowsky and Pnueli [12] (see also [11]).

Like the proof of [10, Theorem 3.3], we proceed by induction on the complexity of a sentence  $\phi \in \text{HEX}^*[\text{FO}_s]$ . The induction step assumes that every well-formed sub-formula of  $\phi$  is logically equivalent to a formula of the desired form and then treats the different ways in which  $\phi$  can be built from its maximal sub-formulae in turn.

Consider the case, in the proof of [10, Theorem 3.3], when  $\phi$  is of the form

$$\forall z \text{TC}[\lambda x, y\psi(x, y)](\mathbf{0}, \mathbf{max}),$$

where  $|x| = |y| = k$ , for some  $k$ , and  $\psi$  is a quantifier-free projection with  $z$  amongst its free variables (but different from those of  $x$  and  $y$ ). Let the underlying signature of  $\psi$  be  $\sigma$  and let  $\mathcal{S}$  be some  $\sigma$ -structure of size  $n$ . The construction in the proof of [10, Theorem 3.3] takes copies of the digraphs  $D_z$  described by  $\psi^{\mathcal{S}}(x, y, z)$ , where the vertices are  $k$ -tuples over  $|S| = \{0, 1, \dots, n-1\}$  and where  $z$  is given a value from  $|S|$ , and strings them together to form the digraph  $D$  by including an edge from the vertex  $\mathbf{max}$  of  $D_z$  to the vertex  $\mathbf{0}$  of  $D_{z+1}$ , for each  $z \in \{0, 1, \dots, n-2\}$ : the vertex  $\mathbf{0}$  of  $D_0$  (respectively  $\mathbf{max}$  of  $D_{n-1}$ ) is denoted as the source (respectively sink) of the resulting digraph  $D$ . Consequently,  $D$  has a path from its source to its sink iff for each  $z \in \{0, 1, \dots, n-1\}$ ,  $D_z$  has a path from (its) vertex  $\mathbf{0}$  to (its) vertex  $\mathbf{max}$ . What is more, it is shown in the proof of [10, Theorem 3.3] that the digraph  $D$  can be described in terms of  $\mathcal{S}$  (uniformly) by a quantifier-free projection so that the source is  $\mathbf{0}$  and the sink is  $\mathbf{max}$  (with the length of these tuples as dictated by the logical description).

When dealing with the operator HEX as opposed to TC, it is not enough to simply repeat the above construction. However, by utilizing the following gadget, Immerman’s construction can be made to work.

Let  $G$  be some (undirected) graph with source  $a$  and sink  $b$ . Let  $\rho(G)$  be obtained from 8 copies of  $G$ , namely  $\{G_i \mid i = 1, 2, \dots, 8\}$ , by amalgamating:

- the sources of  $G_1, G_2, G_3$  and  $G_4$  to form the vertex  $a_1$ ,
- the sources of  $G_5, G_6, G_7$  and  $G_8$  to form the vertex  $a_2$ ,
- the sinks of  $G_1, G_2, G_5$  and  $G_6$  to form the vertex  $b_1$ ,

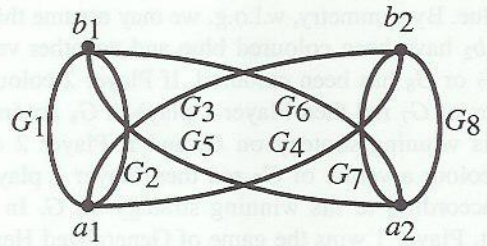


Fig. 1. The graph  $\rho(G)$ .

- the sinks of  $G_3, G_4, G_7$  and  $G_8$  to form the vertex  $b_2$ .

We say that  $\rho(G)$  has two sources,  $a_1$  and  $a_2$ , and two sinks,  $b_1$  and  $b_2$ . The graph  $\rho(G)$  can be visualized as in Fig. 1 where each graph  $G_i$  is represented as a bold line.

By the game of Generalized Hex on  $\rho(G)$  we mean the following. Either Player 1 or Player 2 starts, with both players colouring vertices as usual except that they can also colour the sources and the sinks of  $\rho(G)$ . Player 1 has a winning strategy if he has a strategy which ensures a path of blue-coloured vertices from (at least) one of the sources to (at least) one of the sinks in  $\rho(G)$ : note that the source and the sink must be coloured blue also.

**Lemma 2.** *If Player 1 has a winning strategy in the game of Generalized Hex on  $G$  then Player 1 has a winning strategy in the game of Generalized Hex on  $\rho(G)$  no matter whether Player 1 or Player 2 starts. If Player 2 has a winning strategy in the game of Generalized Hex on  $G$  then Player 2 has a winning strategy in the game of Generalized Hex on  $\rho(G)$  no matter whether Player 1 or Player 2 starts.*

**Proof.** Suppose that Player 1 has a winning strategy in the game of Generalized Hex on  $G$  and that Player 2 starts the game of Generalized Hex on  $\rho(G)$ . By symmetry, w.l.o.g. we may assume that Player 2 plays in the copy  $G_1$  of  $G$  in  $\rho(G)$ .

(a) If Player 2’s first move has coloured the sink  $b_1$  (respectively source  $a_1$ ) red then Player 1 colours the sink  $b_2$  (respectively source  $a_2$ ) blue: by symmetry, w.l.o.g. suppose that  $b_1$  has been coloured red and  $b_2$  has been coloured blue. If Player 2 replies by colouring a vertex of  $G_3$  or  $G_4$  red then Player 1 colours the source  $a_2$  blue; otherwise Player 1 colours the source

$a_1$  blue. By symmetry, w.l.o.g. we may assume that  $a_2$  and  $b_2$  have been coloured blue and no other vertex in  $G_7$  or  $G_8$  has been coloured. If Player 2 colours a vertex of  $G_7$  red then Player 1 plays in  $G_8$  according to his winning strategy on  $G$ , and if Player 2 does not colour a vertex of  $G_7$  red then Player 1 plays in  $G_7$  according to his winning strategy on  $G$ . In any event, Player 1 wins the game of Generalized Hex on  $\rho(G)$ .

(b) If Player 2's first move has coloured a vertex of  $G_1$  but not  $a_1$  or  $b_1$  then Player 1 colours the source  $a_2$  blue. If Player 2 replies by colouring a vertex in  $G_5$  or  $G_6$  then Player 1 colours the sink  $b_2$  blue; otherwise, Player 1 colours the sink  $b_1$  blue. Player 1 then wins as he did in the preceding paragraph.

Reasoning similarly for the case when Player 1 has a winning strategy in the game of Generalized Hex on  $G$  and Player 1 starts the game of Generalized Hex on  $\rho(G)$  yields that Player 1 wins the game of Generalized Hex on  $\rho(G)$ . The second part of the lemma follows similarly.  $\square$

Let  $\psi$  be as in the statement of the theorem and have underlying signature  $\sigma$ . Let  $S$  be a  $\sigma$ -structure of size  $n$  in which we interpret  $\psi$ . Let  $H_z$  be defined as the undirected graph described by  $\psi^S(x, y, z)$ , for each  $z \in \{0, 1, \dots, n-1\}$ , with the vertex  $\mathbf{0}$  (respectively *max*) being the source (respectively sink). Build the graph  $H$  by stringing together the graphs  $\{\rho(H_z) \mid z = 0, 1, \dots, n-1\}$  similarly to as was done above (to obtain the digraph  $D$ ) except by including 4 edges joining both sources of  $\rho(H_z)$  to both sinks of  $\rho(H_{z+1})$ , for  $z = 0, 1, \dots, n-2$ , and amalgamating the two sources (respectively sinks) of  $\rho(H_0)$  (respectively  $\rho(H_{n-1})$ ) to form the source (respectively sink) of  $H$ . (Note that  $H$  is undirected, with one source and one sink, whereas  $D$  is a digraph.) The graph  $H$  can be pictured as in Fig. 2.

Suppose that Player 1 has a winning strategy in the game of Generalized Hex on  $H_z$ , for each  $z \in \{0, 1, \dots, n-1\}$ . By Lemma 2, Player 1 has a winning strategy in the game of Generalized Hex on the graph  $\rho(H_z)$ , for each  $z \in \{0, 1, \dots, n-1\}$ . In the game of Generalized Hex on  $H$ , Player 1's winning strategy is to play according to his winning strategy on each of  $\rho(H_z)$ , for  $z = 0, 1, \dots, n-1$ , as follows. Player 1 begins by playing accord-

ing to his winning strategy on  $\rho(H_0)$  (in fact, any  $\rho(H_z)$  can be adopted as the graph in which Player 1 plays first). In general, if Player 2 plays in  $\rho(H_z)$ , for some  $z \in \{0, 1, \dots, n-1\}$ , then Player 1 replies according to his winning strategy on  $\rho(H_z)$  (note that Player 1 has a winning strategy regardless of whether he plays first or not).

Conversely, suppose that Player 2 has a winning strategy in the game of Generalized Hex on some graph  $H_z$ , for  $z \in \{0, 1, \dots, n-1\}$ . By Lemma 2, Player 2 has a winning strategy in the game of Generalized Hex on the graph  $\rho(H_z)$ . That is, Player 2 has a sequence of moves so that when Player 1 plays first in  $\rho(H_z)$  and no matter how Player 1 plays, Player 1 can not obtain a blue-coloured path from some source to some sink (recall, the source and sink must be coloured blue as well). In the graph  $H$ , Player 2 simply plays this sequence of moves in the subgraph  $\rho(H_z)$  of  $H$ . No matter how Player 1 plays in  $H$ , he will never be able to obtain a blue-coloured path from some source of  $\rho(H_z)$  to some sink of  $\rho(H_z)$ ; and consequently from the source of  $H$  to the sink of  $H$ . Hence, Player 2 has a winning strategy in the game of Generalized Hex on the graph  $H$ .

Consequently, Player 1 has a winning strategy for the game of Generalized Hex on  $H$  iff he has a winning strategy for the game of Generalized Hex on  $\rho(H_z)$ , for each  $z \in \{0, 1, \dots, n-1\}$ .

In general, the construction of  $\rho(G)$  from a graph  $G$ , of size  $n$ , can be described by a quantifier-free projection. The vertices of  $\rho(G)$  are indexed by 4-tuples over  $\{0, 1, \dots, n-1\}$ . Roughly, the first 3 components of the tuple denote which one of the copies  $G_1, G_2, \dots, G_8$  of  $G$  the vertex, given by the 4th component, resides, with  $(0, 0, 0)$  denoting  $G_1$ ,  $(0, 0, n-1)$  denoting  $G_2$ ,  $(0, n-1, 0)$  denoting  $G_3$ , and so on. However, the sources of  $G_1, G_2, G_3$  and  $G_4$  and of  $G_5, G_6, G_7$  and  $G_8$  are amalgamated to form the sources of  $\rho(G)$ , and the sinks of  $G_1, G_2, G_5$  and  $G_6$  and of  $G_3, G_4, G_7$  and  $G_8$  are amalgamated to form the sinks of  $\rho(G)$ . We denote the sources of  $\rho(G)$  by  $(0, 0, 0, 0)$  and  $(n-1, n-1, n-1, 0)$ , i.e., the sources of  $G_1$  and  $G_8$ , and the sinks of  $\rho(G)$  by  $(0, 0, 0, n-1)$  and  $(n-1, n-1, n-1, n-1)$ , i.e., the sinks of  $G_1$  and  $G_8$ . The old sources and sinks of  $G_2, G_3, G_4, G_5, G_6$  and  $G_7$  are left as isolated vertices. The edges of  $\rho(G)$  can clearly be defined by a quantifier-free projection.

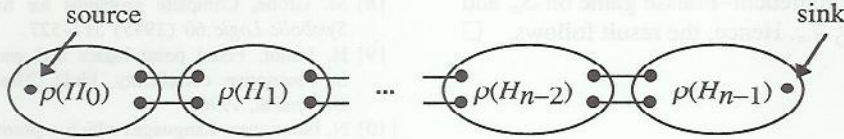


Fig. 2. The graph  $H$ .

For example, the edges in  $G_2$  emanating from a source are defined by the formula

$$x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0 \wedge x_4 = 0$$

$$\wedge y_1 = 0 \wedge y_2 = 0 \wedge y_3 = \max \wedge E(x_4, y_4),$$

and the resulting quantifier-free projection is just a disjunction of similar formula.

Note that when we describe our constructions using logical formula, we often introduce a number of isolated vertices. The addition of isolated vertices to a graph does not make any difference to the winner of the game of Generalized Hex (for both  $G$  or  $\rho(G)$ ).

The description of  $H$  from the graphs  $\{\rho(H_z) \mid z = 0, 1, \dots, n - 1\}$  is then done similarly with an extra component added to the indexing tuples so as to define which copy of  $\rho(H_z)$  a particular vertex belongs to. The source of  $H$  is obtained by amalgamating the two sources of  $\rho(H_0)$ , as above, and calling it  $\theta$ , and the sink of  $H$  is obtained by amalgamating the two sinks of  $\rho(H_{n-1})$ , as above, and calling it  $\max$ . (The reader is referred to, for example, [14] where some quantifier-free projections are given explicitly).

Hence, as the notion of quantifier-free projection is transitive (a result due to Immerman: see Proposition 2.1 of [16]), the problem defined by the sentence

$$\forall z \text{HEX}[\lambda x, y \psi(x, y)](\theta, \max)$$

can be defined by a sentence of the form

$$\text{HEX}[\lambda x', y' \psi'(x', y')](\theta, \max),$$

where  $|x'| = |y'| = k'$ , for some  $k'$ , and  $\psi'$  is a quantifier-free projection.

The other cases for the construction of  $\phi$  in the proof of [10, Theorem 3.3] except with "HEX" replacing "TC" can all be coped with by mimicking Immerman's construction except using the gadget depicted in Fig. 2, as is done above (we leave this as an exercise). Consequently, the result follows.  $\square$

**Corollary 3.** *The problem HEX is complete for PSPACE via quantifier-free projections with successor.  $\square$*

In the absence of a built-in successor relation, a result of Dawar and Grädel [3, Theorem 7.4] tells us that the logic  $(\pm\text{HEX})^*[\text{FO}]$  does not capture PSPACE as it has a 0-1 law. However, this does not rule out a normal form result for  $(\pm\text{HEX})^*[\text{FO}]$  analogous to that in Theorem 1. (For definitions relating to the following theorem, see [17].)

**Proposition 4.** *There are problems definable in  $(\pm\text{HEX})^*[\text{FO}]$  which can not be defined by a sentence of  $\text{HEX}^1[\text{FO}]$  in which the operator HEX does not appear within the scope of the quantifier  $\forall$ .*

**Proof.** By [17, Theorem 14], it suffices to show that for all positive integers  $m$ , there exist  $\sigma_{2^{++}}$ -structures  $S_m$  and  $T_m$  such that

- $S_m \in \text{HEX}$  and  $T_m \notin \text{HEX}$ ,
- $S_m \equiv_{\text{FO}}^m T_m$ ; that is,  $S_m$  and  $T_m$  satisfy exactly the same sentences of quantifier rank at most  $m$ .

Fix  $m$ . Let  $G^l(a, b)$  be the graph with vertex set  $\{a_i, b_i \mid i = 1, 2, \dots, l\} \cup \{a, b\}$  and edge set

$$\{(a_i, a_{i+1}), (a_i, b_{i+1}), (a_{i+1}, b_i), (b_i, b_{i+1}) \mid$$

$$i = 1, 2, \dots, l - 1\} \cup$$

$$\{(a, a_1), (a, b_1), (a_l, b), (b_l, b)\}.$$

The graph  $G^l(a, b)$  is essentially a ladder with diagonal rungs to which the vertices  $a$  and  $b$  are joined at each end. Let  $S_m$  and  $T_m$  each consist of 2 disjoint copies of  $G^l(a, b)$  except that the source of  $S_m$  is vertex  $a$  of the first copy and the sink of  $S_m$  is vertex  $b$  of the first copy, whereas the source of  $T_m$  is vertex  $a$  of the first copy and the sink of  $T_m$  is the vertex  $b$  of the second copy; furthermore, set  $l = 2^m$ .

Clearly,  $S_m \in \text{HEX}$  but  $T_m \notin \text{HEX}$ . A simple induction shows that Duplicator has a winning strategy

in the  $m$ -pebble Ehrenfeucht–Fraïssé game on  $\mathcal{S}_m$  and  $\mathcal{T}_m$ , and so  $\mathcal{S}_m \equiv_{\text{FO}}^m \mathcal{T}_m$ . Hence, the result follows.  $\square$

## References

- [1] S. Abiteboul and V. Vianu, Fixpoint extensions of first order logic and datalog-like languages, in: *Proc. 4th IEEE Symp. on Logic in Computer Science* (1989) 71–79.
- [2] E. Dahlhaus, Skolem normal forms concerning the least fixpoint, in: *Lecture Notes in Computer Science*, Vol. 270 (Springer, Berlin, 1987) 101–106.
- [3] A. Dawar and E. Grädel, Generalized quantifiers and 0–1 laws, in: *Proc. 10th IEEE Ann. Symp. on Logic in Computer Science* (1995) 54–64.
- [4] H.D. Ebbinghaus and J. Flum, *Finite Model Theory* (Springer, Berlin, 1995).
- [5] A. Ehrenfeucht, An application of games to the completeness problem for formalized theories, *Fund. Math.* 49 (1961) 129–141.
- [6] R. Fraïssé, Sur quelques classifications des systèmes de relations, *Publ. Sci. Univ. Alger.* 1 (1954) 35–182.
- [7] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, San Francisco, CA, 1979).
- [8] M. Grohe, Complete problems for fixed-point logics, *J. Symbolic Logic* 60 (1995) 517–527.
- [9] H. Imhof, Fixed point logics and generalized quantifiers in descriptive complexity, Ph.D. Thesis, Albert-Ludwigs Universität, 1996.
- [10] N. Immerman, Languages which capture complexity classes, *SIAM J. Comput.* 16 (1987) 760–778.
- [11] J.A. Makowsky and Y.B. Pnueli, Oracles and quantifiers, in: *Lecture Notes in Computer Science*, Vol. 832 (Springer, Berlin, 1993) 189–222.
- [12] J.A. Makowsky and Y.B. Pnueli, Computable quantifiers and logics over finite structures, in: M. Krynicki, M. Mostowski and L.W. Szczerba, eds., *Quantifiers: Generalizations, Extensions and Variants of Elementary Logic* (Kluwer Academic Publishers, Dordrecht, to appear).
- [13] I.A. Stewart, Comparing the expressibility of languages formed using NP-complete operators, *J. Logic Comput.* 1 (1991) 305–330.
- [14] I.A. Stewart, Methods for proving completeness via logical reductions, *Theoret. Comput. Sci.* 118 (1993) 193–229.
- [15] I.A. Stewart, Context-sensitive transitive closure operators, *Ann. Pure Appl. Logic* 66 (1994) 277–301.
- [16] I.A. Stewart, Completeness of path problems via logical reductions, *Inform. and Comput.* 121 (1995) 123–134.
- [17] I.A. Stewart, Logics with zero-one laws that are not fragments of bounded-variable infinitary logic, *Math. Logic Quart.*, to appear.

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