The Greedy Algorithm and the Cohen-Macaulay property of rings, graphs and toric projective curves

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Abstract It is shown in this paper how a solution for a combinatorial problem obtained from applying the greedy algorithm is guaranteed to be optimal for those instances of the problem that, under an appropriate algebraic representation, satisfy the Cohen-Macaulay property known for rings and modules in Commutative Algebra. The choice of representation for the instances of a given combinatorial problem is fundamental for recognizing the Cohen-Macaulay property. Departing from an exposition of the general framework of simplicial complexes and their associated Stanley-Reisner ideals, wherein the Cohen-Macaulay property is formally defined, a review of other equivalent frameworks more suitable for graphs or arithmetical problems will follow. In the case of graph problems a better framework to use is the edge ideal of Rafael Villarreal. For arithmetic problems it is appropriate to work within the semigroup viewpoint of toric geometry developed by Antonio Campillo and collaborators.

1 Introduction

A greedy algorithm is one of the simplest strategies to solve an optimization problem. It is based on a step-by-step selection of a candidate solution that seems best at the moment, that is a local optimal solution, in the hope that in the end this process leads to a global optimal solution. Greedy algorithms do not always output (global) optimal solutions, but for many optimization problems they do.

For Antonio Campillo and Miguel Angel Revilla.

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A seminal result of Jack Edmonds [Edmonds, 1971], states that for a greedy algorithm to output optimal solutions it is necessary and sufficient that the input is a *matroid*, regardless of the weight function associated to the optimization problem. A matroid is a simplicial complex with the additional *exchange property*. Korte and Lovazs have extended Edmonds result to cover other type of weight functions and weaker combinatorial objects as input. Specifically they have slimmed the matroid by removing the subclusiveness property and named the new object a *greedoid* [Korte & Lovász, 1984].

The intuition behind Edmonds result is to view the problem of determining the correctness of the greedy algorithm as a localization problem: in order to know if greedy works for some set H, it suffices to look at some discrete partition of H and check if greedy works for each of the parts then it should work for H. This is a classical working paradigm for the algebraic geometer (see, e.g., [Kunz, 1985]), namely to solve problems locally and translate solutions globally, and vice versa. Matroids fit in very well this local–global pattern, which is best viewed in the realm of Commutative Algebra as follows:

> As a simplicial complex Δ over some ground set S, a matroid is such that for every subset W of S, the induced subcomplex $\Delta_W := \{F \in \Delta : F \subseteq W\}$ has an associated ideal in some ring of polynomials which is Cohen-Macaulay. By extension we say then that each Δ_W is Cohen-Macaulay [Stanley, 1996].

This suggests that in order to guarantee optimal solutions from the greedy algorithm we should start with those instances of the input that are Cohen-Macaulay or piece–wise Cohen-Macaulay (as matroids).

Now, a related question is how to recognize those Cohen-Macaulay instances for a given combinatorial problem. This is a question about the choice of representation, since there is more than one way to associate to a simplicial complex some module that encodes its algebraic properties, like being Cohen-Macaulay. It also has to do with the chosen ground set of the simplicial complexes. The selection of the type of simplicial complex and associated module should be determined by the type of combinatorial problem to which we apply some form of greedy algorithm.

For a general ground set S identified with an initial segment of the natural numbers, the standard module to associate is the Stanley-Reisner ring over some ring of polynomials [Stanley, 1996]. However, for problems on graphs there is a natural ideal to associate, which is the *edge ideal* [Villarreal, 1990], equivalent to a Stanley-Reisner ideal over a particular simplicial complex whose faces correspond to the independent sets of the graph. For arithmetic problems, an alternative framework is given by the numeric semigroups viewpoint of Toric Geometry [Campillo & Pisón, 2001, Campillo & Gimenez, 2000], which serves as a bridge between affine and projective toric varieties to and from polytopes and simplicial complexes. We will illustrate in the following sections the usefulness of these algebraic ideas for ascertaining the correctness of greedy algorithms.

2 Greedy algorithms and simplicial complexes

Let S be a finite set and Δ a collection of subsets of S. We say that (S,Δ) is a *simplicial complex* if it verifies the following two conditions:

(S1) For all $s \in S$, $\{s\} \in \Delta$.
(S2) $F \subseteq G \in \Delta$ implies $F \in \Delta$.

S is called the *ground* set and is usually identified with $[n] := \{1,\ldots,n\}$, in which case one refers to the simplicial complex as Δ. The elements of Δ are called *faces*, and the maximal elements of Δ, with respect to $\subseteq$, are called *facets* (or Δ-maximal sets). The dimension of the simplicial complex Δ, denoted $\dim(\Delta)$, is the maximum dimension of its faces, where the dimension of face F is $\dim(F) = |F|-1$, where $|F|$ denotes the cardinality of F.

Given a simplicial complex Δ over the ground set S, and given a *linear* weight function $f : S \to \mathbb{R}_{\geq 0}$, we can extend f to 2^S (the set of subsets of S) by defining for each $A \subseteq S$, $f(A) = \sum_{a\in A} f(a)$. We state the general form of an optimization problem as a maximization problem.

Definition 1. The Optimization Problem for (S,Δ) and weight function $f : S \to \mathbb{R}_{\geq 0}$, denoted f-OPT(Δ), is the following:

To find a set $A \in \Delta$ with maximum f-weight.

Observe that if f is linear, or at least *monotone* (i.e. $A \subseteq B$ implies $f(A) \leq f(B)$), then the optimization problem f-OPT(Δ) reads:

To find a facet of Δ with maximum f-weight.

From now on we assume that f is linear.

Definition 2. The greedy algorithm associated to Optimization Problem for (S,Δ) and weight f, denoted GREEDY, is presented in Figure 1.

```
GREEDY
Input: (S,Δ) and f : S → ℝ≥0
A ← ∅
 sort S in nonincreasing order by weight f
 while S ≠ ∅ do
     choose a ∈ S in the nonincreasing order by f
     S ← S − {a}
     if A ∪ {a} ∈ Δ then A ← A ∪ {a}
 end while
end
Output: A.
```

Fig. 1 Algorithm GREEDY.

Remark 1. Monotonicity (or positive linearity) of f is needed for inducing a partial order in S and sorting makes sense. The algorithm always terminates because S is finite, and it will always output a non empty set A which is contained in a facet. The complexity of GREEDY will mostly depend on the membership test in line 6. Note that f can be a positive constant function. In this case we can select the elements in S in any order, and the correctness of GREEDY have to be determined for any of the possible orders of selecting the equally weighted elements. Apart from this difficulty, the consideration of constant f is useful to treat under the GREEDY scheme optimization problems where instances are not explicitly weighted and we turn them into weighted problems by assigning equal constant weight to every element. This we will do in Section 4.

Definition 3. We say that the algorithm GREEDY for (S,Δ) and f *correctly solves* the associated optimization problem if it gives as output a set A such that: (*i*) A is Δ-maximal (a facet) and (*ii*) for all $B \in \Delta$ $(f(A) \geq f(B))$.

We begin by showing that we need no extra assumptions about Δ (other than to be a simplicial complex) to guarantee that the output of GREEDY is a Δ-maximal set.

Proposition 1. *The output of* GREEDY *is a Δ-maximal set.*

Proof. Let A be the output and suppose A is not maximal. Then there is a $C \in \Delta$ such that $A \subset C$. Let $x \in C - A$, then $A \cup \{x\} \subseteq C \in \Delta$, and hence $A \cup \{x\} \in \Delta$. But this x must have been considered at some step in the algorithm and should have been placed in A, so $x \in A$, a contradiction. □

Since the previous result holds regardless of the weight function f, what then we really need to guarantee is that

$$\boxed{\text{GREEDY outputs an } A \text{ of maximum } f\text{-weight}}$$

For some weight functions (e.g., constant functions), one way to achieve this is to impose on Δ the stronger condition of being *pure*, that is

$$\boxed{\text{all } \Delta\text{-maximal elements have same dimension}}$$

For a more ample spectrum of weight functions (e.g. linear), GREEDY achieves optimal solutions for inputs where the pureness condition can be localized. This is the point of matroids, and by extension of Cohen-Macaulay complexes.

3 Matroids and Cohen-Macaulay complexes

A *matroid* is a simplicial complex (S,Δ) which, in addition, verifies the following principle:

$\boxed{\text{\textbf{Principle of Exchange (PE):} If } A,B \in \Delta \text{ and } |A| < |B| \text{ then there exists } x \in B - A \text{ such that } A \cup \{x\} \in \Delta.}$

The Principle of Exchange is equivalent to a *localization* of the property of being pure[1]:

$$\boxed{\text{PE} \iff \forall W \subseteq S,\ \Delta_W := \{F \in \Delta : F \subseteq W\} \text{ is pure}}$$

Moreover, the Principle of Exchange is equivalent to the correctness of GREEDY. This is Edmond's result on matroids and the greedy algorithm [Edmonds, 1971], but see [Papadimitriou & Steiglitz, 1998, Theorem 12.5, p. 285] for a textbook exposition of this important result in optimization. We collect all these facts in the following theorem (and using an updated notation and terminology from Commutative Algebra).

Theorem 1 ([Edmonds, 1971]). *Given a simplicial complex Δ over a ground set S, the following statements are equivalent:*

(i) GREEDY *correctly solves the optimization problem for (S,Δ) and any (linear) weight function.*
(ii) The Principle of Exchange (i.e. (S,Δ) is a matroid).
(iii) $\forall W \subseteq S$, the induced subcomplex Δ_W is pure. $\square$

We shall see next that the localization of pureness is equivalent to a localization of the Cohen-Macaulay property. Thus, matroids are locally Cohen-Macaulay complexes, and we can view the correctness of GREEDY in the world of Cohen-Macaulay rings, and extensions, which we argue here to be an appropriate algebraic framework (if not the correct one) to understand the workings of GREEDY.

3.1 Greedy on locally Cohen-Macaulay complexes

Given a simplicial complex Δ over the ground set $[n] := \{1,\dots,n\}$, and given $\mathscr{K} := k[x_1,\dots,x_n]$, the polynomial ring in n variables over some field k, the Stanley–Reisner ideal of Δ is the square free monomial ideal $I_\Delta = \langle \mathbf{x}^A : A \notin \Delta \rangle \subseteq \mathscr{K}$, where $\mathbf{x}^A := x_{i_1}x_{i_2}\cdots x_{i_r}$ with $A = \{i_1,\dots,i_r\} \subseteq [n]$. The Stanley-Reisner ring (or *face ring*) is the quotient $R_\Delta = \mathscr{K}/I_\Delta$. From the correspondence between Δ and I_Δ, the latter can be characterized as

$$I_\Delta = \bigcap_{\substack{F\in\Delta \\ F \text{ a facet}}} M^{F^c} \tag{1}$$

where M^{F^c} is the monomial prime ideal corresponding to the non-face $F^c = [n] \setminus F$; in other words, $M^{F^c} = \langle x_i : i \notin F \rangle$. Equation (1) gives an *irreducible decomposition* of I_Δ; as M^{F^c}, for F a facet (a maximal face), is an *irreducible component* provided it is not redundant (i.e. cannot be deleted).

[1] The reader is encouraged to prove this equivalence.

The dimension of the face ring, $\dim(R_\Delta)$, can be defined as $\dim(R_\Delta) = \dim(\Delta) + 1$, and therefore[2]

$$\dim(R_\Delta) = \max[|F| : F \in \Delta] \tag{2}$$

The codimension of R_Δ, $\mathrm{codim}(R_\Delta)$, can be defined as the *smallest number of generators of any irreducible component of* I_Δ (see [Miller & Sturmfels, 2005, §5.5]). Other two useful measures of dimension are: 1) the *projective dimension* of R_Δ, $\mathrm{pd}(R_\Delta)$, as the length of a minimal resolution of R_Δ; and 2) the *depth* of R_Δ, $\mathrm{depth}(R_\Delta)$, as the maximal length of a regular sequence on R_Δ. All these forms of dimension are related as follows:

$$\mathrm{pd}(R_\Delta) \geq \mathrm{codim}(R_\Delta) \text{ and } \dim(R_\Delta) \geq \mathrm{depth}(R_\Delta).$$

Now, the ideal I_Δ (or equivalently the ring R_Δ) is *Cohen-Macaulay* if and only if $\mathrm{pd}(R_\Delta) = \mathrm{codim}(R_\Delta)$ if and only if $\mathrm{depth}(R_\Delta) = \dim(R_\Delta)$.

The simplicial complex Δ is Cohen-Macaulay if its face ring R_Δ is Cohen-Macaulay.

Remark 2. Technically the Cohen-Macaulay property depends on the choice of the field k, because computing regular sequences involves finding non zero divisors of certain quotient modules, and being a divisor or not depends on the characteristic of the field. Hence, we will always assume that our field k is of characteristic 0. For further simplicity the reader can assume that k is the field of real numbers.

A consequence of the above definitions and facts about the Stanley-Reisner ring R_Δ is the following result (cf. [Bruns & Herzog, 1993, Corollary 5.1.5] or [Miller & Sturmfels, 2005, p. 114]):

Proposition 2. *If the face ring $\mathscr{K}/I_\Delta$ is Cohen-Macaulay then all irreducible components of I_Δ have equal cardinality.* □

It follows from the above result that a Cohen-Macaulay simplicial complex is pure. The converse is true locally, that is, a locally pure complex (i.e. a matroid) is locally Cohen-Macaulay [Stanley, 1996, Proposition 3.1].

We then have the following characterization of the correctness of the greedy algorithm in terms of the Cohen-Macaulay property:

Theorem 2. *Given a simplicial complex Δ over a ground set S, the following statements are equivalent:*

(i) GREEDY *correctly solves the optimization problem for (S,Δ) and any (linear) weight function.*
(ii) $\forall W \subseteq S$, Δ_W is pure (i.e. (S,Δ) is a matroid).
(iii) $\forall W \subseteq S$, Δ_W is Cohen-Macaulay.

Proof. The equivalence of (i) and (ii) is Theorem 1, and the equivalence of (ii) and (iii) is Proposition 3.1 of [Stanley, 1996]. □

[2] The dimension of a finitely generated ring is the maximum cardinality of an algebraically independent set. This is equivalent in R_Δ to $\dim(\Delta) + 1$ and Eq. (2).

4 Maximum Independent Set and Cohen-Macaulay graphs

The Maximum Independent Set problem (or MIS) is the optimization problem that asks for a largest subset of vertices in a graph which are pairwise non-adjacent. A dual problem to MIS is the Minimum Vertex Cover (or MVC), which asks for the smallest subset of vertices in a graph where all the edges have at least one endpoint. Both problems are related to each other by the following equivalence: *a set of vertices is a vertex cover if, and only if, its complement is an independent set*. The MIS (and the MVC) problem is **NP**-complete [Garey & Johnson, 1979], and in view of its general intractability various greedy algorithms have been proposed for obtaining approximate solutions. A common feature of many of these greedy strategies for finding solutions to the MIS is to select vertices in some order with respect to their degrees (i.e. number of incident edges) and remove them and their adjacent vertices at each step. Although, in general, these strategies based on vertex selection have a poor approximation ratio under a worst case analysis (cf. [Papadimitriou & Steiglitz, 1998, §17.1] or [Dinur & Safra, 2005]), some of these work for some classes of graphs, meaning that they do provide us with the optimal solution (an independent set of maximum possible cardinality). We shall see that these vertex selection greedy strategies (dependable on the order of selecting the vertices) work in general for Cohen-Macaulay graphs.

First we shall fix some notation and terminology on graphs.

Definition 4. We denote graphs as $G = \langle V(G),\, E(G)\rangle$ or $\langle [n],\, E(G)\rangle$, where the vertex set $V(G)$ of cardinality n is identified with the labelling set $[n] := \{1,2,\dots,n\}$, and $E(G) \subseteq \{\{i,j\} : i,j \in V(G)\}$ is the set of edges. The complement of G, denoted G^c, is a graph with same set of vertices as G and edge set $E(G^c) := \{\{i,j\} : \{i,j\} \notin E(G)\}$. Given a vertex $a \in V(G)$, the neighbourhood of a is the set $N_G(a) = \{b : \{a,b\} \in E(G)\}$, and its degree is denoted $\deg(a)$. For a subset of vertices $W \subseteq V(G)$, the graph induced by W has as vertices the set W and as edges all those in $E(G)$ among pairs of vertices in W. The graph induced by W is formally denoted G_W.

All throughout this paper a graph is always simple and undirected. Given a graph G, a subset C of $V(G)$ is a *clique* if for all distinct pairs $i,j \in C$, $\{i,j\} \in E(G)$. An independent set of G is a subset $M \subseteq V(G)$ such that for all $i,j \in M$, $\{i,j\} \notin E(G)$. The independent set M is *maximal* if no extension of M is an independent set. A *maximum* independent set (MIS) is a (maximal) independent set of greatest possible cardinality. A vertex cover of G is a subset $C \subseteq V(G)$ such that for all $\{i,j\} \in E(G)$, $C \cap \{i,j\} \neq \emptyset$. C is *minimal* if no proper subset of C is a vertex cover of G, and is a *minimum* vertex cover (MVC) if it has smallest possible cardinality. It is usually said that G is *unmixed* if all of its minimal vertex covers have the same size.

Next, we shall denote *the vertex selection strategy following the rule* ω *for selecting vertices* as $\textsc{VertexSelect}[\omega]$, and which proceeds as follows:

$\textsc{VertexSelect}[\omega]$: select a vertex in the order established by the rule ω, remove its neighbours and repeat the selection procedure in the reduced graph. The output is the set of selected vertices.

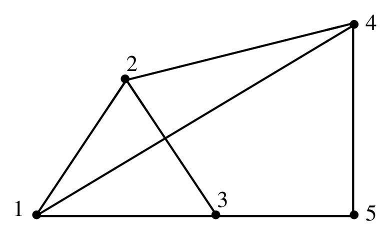
By construction the output of VERTEXSELECT$[\omega]$ is a maximal independent set. We shall first deal with two simple rules for selecting vertices:

ω_L: choose the vertex with largest degree first;
ω_S: choose the vertex with smallest degree first.

Note that for both rules, selection of vertices is always possible, so both variants of the VERTEXSELECT$[\omega]$ algorithm terminate. Also, under these rules, vertices are sorted in the order given by their degrees, breaking ties at each step of the algorithm by using the implicit order given initially by the numeric labelling of the vertices.

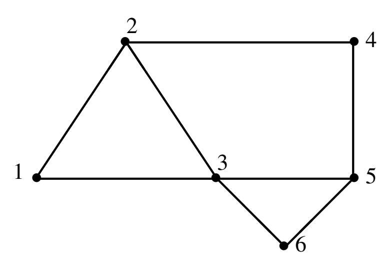
An example where the algorithm VERTEXSELECT$[\omega]$ outputs an optimal solution, regardless of ω, is the graph in Figure 2. For this graph an optimal output obtained using the rule ω_S of the smallest degree first is the pair $\{5,1\}$ but it could also be: $\{2,5\}$, or $\{3,4\}$, where the last two pairs can also be obtained with the rule ω_L of the largest degree first.

Fig. 2 A graph for which VERTEXSELECT$[\omega]$ gives optimal solution for $\omega \in \{\omega_S, \omega_L\}$.



On the other hand, Figure 3 shows a graph where VERTEXSELECT$[\omega_S]$ gives the optimal solution $\{1,4,6\}$, whilst VERTEXSELECT$[\omega_L]$ gives $\{3,4\}$, an independent set which is not maximum.

Fig. 3 A graph for which VERTEXSELECT$[\omega_S]$ gives optimal solution, but VERTEXSELECT$[\omega_L]$ does not.



We will see next how to associate a suitable simplicial complex to a graph in order to analyze the correctness of VERTEXSELECT$[\omega]$ through the lens of Commutative Algebra.

4.1 The Edge Ideal and correctness of VERTEXSELECT

Let G be a graph on $[n]$ and edge set $E(G)$; let $\mathscr{K} = k[x_1,\dots,x_n]$, with k a field. To G one can associate the *edge ideal* $I(G) \subseteq \mathscr{K}$, which is generated by all square–free monomials $x_i x_j$ with $(i,j) \in E(G)$. This edge ideal seems to have been first defined by Rafael Villarreal in [Villarreal, 1990].

To the complementary graph of G, G^c, one can associate the *clique complex* $\kappa(G^c)$ where a face of dimension d is a clique of G^c of size $d+1$; that is, any subset $A \subseteq [n]$ is in $\kappa(G^c)$ iff $\forall i \forall j (i,j \in A \to (i,j) \in E(G^c))$.

Hence, the Stanley–Reisner monomial ideal associated to $\kappa(G^c)$, namely $I_{\kappa(G^c)}$, is exactly the edge ideal $I(G)$, and the following definition is sound:

Definition 5. A graph $G = \langle [n], E(G) \rangle$ is Cohen-Macaulay over a field k, if the edge ideal $I(G) = \langle x_i x_j : (i,j) \in E(G) \rangle$ (and equivalently, the ideal $I_{\kappa(G^c)}$) is Cohen-Macaulay over $\mathscr{K} = k[x_1,\dots,x_n]$, that is, $\mathscr{K}/I(G)$ (or $\mathscr{K}/I_{\kappa(G^c)}$) is Cohen-Macaulay ring.

Cohen-Macaulay graphs are extensively studied in [Villarreal, 1990]. For example, the only cycles that are Cohen-Macaulay are those of three or five vertices.

Now, if $F \subset [n]$ is a maximal clique in G^c then F is a maximal *independent set* in G and $F^c := [n] \setminus F$ is a minimal *vertex cover* in G of cardinality $n - |F|$, and vice versa. From this combinatorial equivalence we can derive the following facts:

Fact 1: Given a graph G, another simplicial complex often associated to G is $\Delta(G)$ consisting of all independent sets of G. By the previous equivalence one sees that $\Delta(G) = \kappa(G^c)$. Thus, the edge ideal $I(G) = I_{\Delta(G)}$.

Fact 2: By Equation (1) the irreducible components of the edge ideal $I(G)$ are obtained from the intersection of monomial prime ideals of the form $\langle x_i : i \in C \rangle$ such that C is a minimal vertex cover of G.

Fact 3: Using Equation (2), we get that the dimension of the face ring of $\Delta(G)$, $\dim(R_{\Delta(G)}) = \dim(\mathscr{K}/I(G))$, is equal to the maximum cardinality of a clique in the complementary graph G^c, or equivalently, the maximum cardinality of an independent set in G. On the other hand, $\mathrm{codim}(R_{\Delta(G)})$ is equal to the minimum cardinality of a vertex cover in G. This is by definition of codimension and simply noting that (max. cardinality of independent set in G) + (min. cardinality of vertex cover in G) = n.

Theorem 3. *If the graph G is Cohen-Macaulay then, for any of the degree-based rules $\omega \in \{\omega_S, \omega_L\}$, the algorithm* VERTEXSELECT$[\omega]$ *outputs an optimal solution on input G.*

Proof. By Proposition 3 all irreducible components of $I(G) = I_{\Delta(G)}$ have same cardinality, which by Fact 2 means that G is unmixed, and since VERTEXSELECT$[\omega]$ always output a maximal solution, this solution is optimal. □

The converse of Theorem 3 is not true. We have seen that for the graph G in Figure 2 the algorithm VERTEXSELECT[ω] outputs an optimal solution, for either rule ω_S or ω_L, because G is unmixed. However, this graph G is not Cohen-Macaulay: the dimension of its face ring, $\dim(k[x_1,\dots,x_5]/I(G)) = 2$, but its depth is $\mathrm{depth}(k[x_1,\dots,x_5]/I(G)) = 1$.

We can use Theorem 3 to produce a polynomial-time deterministic test for the failure of the Cohen-Macaulay property in graphs.

An algorithmic criterion for showing that a graph is not Cohen-Macaulay. Given a graph G as input, run VERTEXSELECT[ω] for both rules ω_S and ω_L, and check that the outputs are of different cardinalities.

By this criterion the graph shown in Figure 3 is not Cohen-Macaulay.

Moreover, it is instructive to see how VERTEXSELECT[ω] fits in the general greedy scheme (Definition 2). Take as weights on the set of vertices a counting function, i.e. $f : V(G) \to \mathbb{R}$ given by $f(v) = 1$. Then VERTEXSELECT[ω] is the GREEDY algorithm with this counting function, and ω_S and ω_L give particular orderings for selecting the vertices (cf. Remark 1). In fact, for the graph in Figure 2 it does not matter in which order the vertices are taken, GREEDY will always give the optimal independent set (of size 2) *for this particular weight function*. However, GREEDY will fail to give the maximum weighted, and maximum independent, set of vertices if the weights are given by $f(v) = \deg(v)$ and now we select vertices ordered from highest to lowest degree (breaking ties by using the order given by their labelling). For this weight f, in the graph of Figure 2, we get as solution $\{1,5\}$ whose deg-weight is $3+2 = 5$, whilst the MIS of maximum weight of 6 is $\{3,4\}$. For the graph in Figure 3 and $f(v) = \deg(v)$, the greedy solution is $\{3,4\}$, which has maximum weight of 6, but is not a MIS. The MIS of maximum weight in this case is $\{1,4,6\}$. By Theorem 2, it follows that neither of these graphs is a matroid. This can also be seen by observing that the subgraph induced by $W = \{3,4,5\}$ has as set of facets $\{\{3,4\},\{5\}\}$, so it is not pure.

5 Chordal graphs and Shellable simplicial complexes

A special subclass of Cohen-Macaulay complexes are the shellable simplicial complexes [Stanley, 1996]. The associated edge ideal allows to refine our analysis of correctness of greedy algorithms for the MIS problem for more interesting classes of graphs, like the chordal graphs.

Definition 6 (Shellable simplicial complex). A simplicial complex Δ is called *shellable* if the facets of Δ can be arranged in a linear order $F_1, \dots, F_m$ such that for each pair i, j, $1 \le i < j \le m$, there exists some $v \in F_j \setminus F_i$ and some $k < j$ such that $F_j \setminus F_k = \{v\}$. Such an ordering of the facets is called a *shelling order*.

A pure shellable simplicial complex is Cohen-Macaulay [Bruns & Herzog, 1993, Theorem 5.1.13]. The notion of shellability (as in the previous definition) can also be applied to non pure complexes, as it is done in [Björner & Wachs, 1996].

A graph G is *shellable* if the simplicial complex $\Delta(G)$ is shellable (in the non pure sense). The notion of shellable graph was introduced and studied in [van Tuyl & Villarreal, 2008], and the remarkable main result contained in that paper is that *chordal graphs are shellable*.

Theorem 4 ([van Tuyl & Villarreal, 2008]). *If G is a chordal graph then G is shellable.* $\square$

Recall that a graph is *chordal* if every cycle in it of length > 3 has a *chord*, where a chord is an edge between two non-consecutive vertices. As an example, drawing an edge among vertices 3 and 4 in the graph of Figure 3 makes that graph chordal. Observe that with that extra edge, applying $\text{VERTEXSELECT}[\omega_S]$ to the extended graph still gives the optimal solution $\{1,4,6\}$, whilst $\text{VERTEXSELECT}[\omega_L]$ gives $\{3\}$, so by our Theorem 3 this chordal graph is not Cohen-Macaulay.

We seek to answer the following two questions for chordal graphs:

1. What version of our $\text{VERTEXSELECT}[\omega]$ algorithm may work correctly for chordal graphs?
2. How to algebraically characterize the correctness of $\text{VERTEXSELECT}[\omega]$?

The answer to question 1 is found implicitly in a combinatorial characterization of chordality due to Fulkerson and Gross [Fulkerson & Gross, 1965]:

Fulkerson-Gross criterion: Given a graph G, to know if it is chordal, it is necessary and sufficient that the following procedure eliminates all the vertices of G: search for a simplicial vertex in G, and if one is found, suppress it and repeat the procedure in the reduced graph.

Recall that a vertex v is simplicial if the subgraph induced by v and its neighbours, $\{v\} \cup N(v)$, form a clique. Observe that the Fulkerson-Gross simplicial vertices selection procedure establishes an ordering of the vertices, which has been named a *perfect elimination ordering*. A perfect elimination ordering of a graph is an ordering of its vertices such that each vertex in the ordering and its neighbours following it in the order form a clique. The Fulkerson-Gross criterion translates then to the following statement:

A graph G is chordal iff G has a perfect elimination ordering

Now, let us denote by ω_{pe} a perfect elimination ordering $v_1, \ldots, v_n$ of the vertices, and let $\text{VERTEXSELECT}[\omega_{pe}]$ be VERTEXSELECT with the Fulkerson-Gross selection strategy consisting in selecting a simplicial vertex (if it exists), remove its neighbours and repeat the procedure in the reduced graph, until no simplicial vertex is found or there are no more vertices to select. The collected vertices this way form an independent set, but observe that a priori there is no guarantee that it will be a maximal independent set because the algorithm may stop at some step where it does not finds any more simplicial vertices. Of course it will be guaranteed to end with a solution to MIS if the input graph is chordal. Thus we have an answer to question 1 in the following theorem.

Theorem 5. $\text{VERTEXSELECT}[\omega_{pe}]$ *correctly solves the MIS problem for input graph G if and only if G is chordal.* $\square$

Back to our graph in Figure 3 with the extra edge $\{3,4\}$ to turn it chordal, $\text{VERTEXSELECT}[\omega_{pe}]$ on input this graph will output the maximum independent set $\{1,4,6\}$.

There is more to say about chordal graphs in Commutative Algebra. A celebrated theorem by Fröberg [Fröberg, 1990] states that:

The edge ideal $I(G)$ of a graph G has a linear resolution if and only if its complementary graph G^c is chordal.

On the other hand, the following is known (cf. [Miller & Sturmfels, 2005]):

If an edge ideal $I(G)$ of a graph G has linear quotients, then $I(G)$ has a linear resolution.

The missing ingredient to answer question 2 was found in the recent work [Guo, Shen & Wu, 2016, Guo, Shen & Wu, 2017], where the new notion of strongly shellable is introduced.

Definition 7 (Strongly shellable [Guo, Shen & Wu, 2017]). A simplicial complex Δ is called *strongly shellable* if its facets can be arranged in a linear order $F_1, \dots, F_m$ in such a way that for each pair $i < j$, there exists $k < j$, such that $|F_j \setminus F_k| = 1$ and $F_i \cap F_j \subseteq F_k \subseteq F_i \cup F_j$. Such an ordering of facets is called a *strong shelling order*.

By definition strongly shellable simplicial complexes are shellable. Less obvious is the fact that matroids are (pure) strongly shellable [Guo, Shen & Wu, 2016].

Now, for a finite simple graph G, with vertex set $V(G)$ and edge set $E(G)$, a strong shelling order $\succ$ on the edge set $E(G)$ (corresponding to a strong shelling order of the edge ideal $I(G)$), means that whenever we have two disjoint edges $E_i \succ E_j$, then we can find some $E_k \succ E_j$ that intersects both E_i and E_j non-trivially. If such strong shelling order on $E(G)$ exists, one says that G is *edgewise strongly shellable*.

Putting together the known facts on chordal graphs, linear quotients and linear resolutions of edge ideals, with this new combinatorial property of strongly shellable, Guo, Shen and Wu have shown the following beautiful result:

Theorem 6 ([Guo, Shen & Wu, 2017]). *Let G be a finite simple graph. Then the following conditions are equivalent:*

(i) G is edgewise strongly shellable.
(ii) The edge ideal $I(G)$ has linear quotients.
(iii) The edge ideal $I(G)$ has a linear resolution.
(iv) The complement graph G^c is chordal.
(v) The complement graph G^c has a perfect elimination ordering. $\square$

Turning this around our greedy (algorithmic) interests, we obtain the following answer to question 2:

Theorem 7. *Let G be a finite simple graph. Then the following conditions are equivalent:*

(i) VERTEXSELECT$[\omega_{pe}]$ *is correct on G.*
(ii) G is chordal.
(iii) G has a perfect elimination ordering.
(iv) The complement graph G^c *is edgewise strongly shellable.*
(v) The edge ideal of the complement graph $I(G^c)$ *has linear quotients.*
(vi) The edge ideal of the complement graph $I(G^c)$ *has a linear resolution.* □

There is certainly much more to research on the correctness of greedy algorithms for the MIS problem on other classes of graphs. Take for example the *Petersen* graph. This graph is not chordal; VERTEXSELECT$[\omega]$ with ordering by degrees gives different sizes of independent sets; hence it is not Cohen-Macaulay.

6 Coin-Exchange problems and Cohen-Macaulay Toric Projective Curves

Given a set of coin values $M = \{e_1, \dots, e_h\}$ and a target value $B > 0$, the Coin-Exchange problem asks for the minimum number of coins whose values sum up to B. We will always assume that $e_1 = 1$ so that all values can be attained, and that $e_1 < e_2 < \dots < e_h$. Thus, formally, the Coin-Exchange problem (a.k.a. *optimal representation problem*) reads:

Input: Integers $1 = e_1 < e_2 < \dots < e_h$, and $B > 0$

Output: h-tuple of non negative integers $(a_1, a_2, \dots, a_h)$ such that

$B = \sum_{i=1}^{h} a_i e_i$ and $\sum_{i=1}^{h} a_i$ is minimized.

The h-tuple $(a_1, a_2, \dots, a_h)$ is called a representation of B, and the one that minimizes $\sum_{i=1}^{h} a_i$ is called an optimal representation, and we write $\mathrm{opt}(B; e_1, e_2, \dots, e_h)$ for the minimum value $\sum_{i=1}^{h} a_i$. If the coin system is clear from the context we might just write $\mathrm{opt}(B)$.

The Coin-Exchange problem is a form of the Knapsack problem, and so it is in general NP-complete (cf. Integer Knapsack in [Garey & Johnson, 1979]). This justifies the design of heuristics for obtaining approximate solutions, that for some cases may find the optimal. One such heuristics is the following greedy algorithm.

Greedy strategy for optimal Coin-Exchange. A greedy strategy to solve the optimal representation problem (or Coin-Exchange) works as follows: Initially consider an empty h-tuple α. Then, repeat the following steps until $B = 0$: find the largest index i such that coin $e_i \le B$, add 1 in the ith entry of α, and replace B by $B - e_i$.
The output $\alpha := (a_1, a_2, \dots, a_h)$ obtained with this greedy strategy will be called the *greedy representation* of B, and write $\mathrm{greed}(B; e_1, e_2, \dots, e_h)$, or simply $\mathrm{greed}(B)$, for the value $\sum_{i=1}^{h} a_i$ obtained by this algorithm.

Throughout the rest of this section the "greedy algorithm" refers to the above greedy strategy for the optimal Coin-Exchange problem.

Here is a list of some obvious facts that will be useful later:

(i) For a set of two coins $\{1, e_2\}$, $\mathrm{greed}(B) = \mathrm{opt}(B)$ for all B.
(ii) For any coin value $e_j \in \{e_1, e_2, \ldots, e_h\}$ and for every $B \geq e_j$,

$$\mathrm{opt}(B) \leq \mathrm{opt}(B - e_j) + 1, \tag{3}$$

with equality if coin e_j is used in a minimal representation of B. (Since $B = B - e_j + e_j$, an optimal representation of $B - e_j$ gives a representation of B adding 1 to the jth term.)
(iii) The value of the greedy representation bounds the optimal value for the Coin-Exchange:

$$\mathrm{greed}(B; e_1, e_2, \ldots, e_h) \geq \mathrm{opt}(B; e_1, e_2, \ldots, e_h) \tag{4}$$

We can assume we have the following implementation of the greedy algorithm: for each $i = h, h-1, \ldots, 2, 1$, let $a_i = \lfloor B/e_i \rfloor$, and set B to be $B - e_i a_i$. This produces the greedy representation in time $O(h \log e_h)$. This is polynomial in h. (Observe that we could have $e_h = 2^h$.)

An example for which the greedy algorithm fails to output an optimal representation of the Coin-Exchange problem is given by the system $\{1, 3, 4\}$ and the target $B = 6$. The greedy representation for this value is $(2, 0, 1)$ of size $\mathrm{greed}(6) = 3$, while the optimal representation is $(0, 2, 0)$ of size $\mathrm{opt}(6) = 2$. On the other hand, for the US coin system (of cents) $\{1, 5, 10, 25, 50, 100\}$ or the Eurozone coin system $\{1, 2, 5, 10, 20, 50, 100, 200\}$, the greedy algorithm will always produce an optimal representation for any given value. A formal proof of these facts will be possible after we endowed this numerical problem with the algebraic structure appropriate for applying the theory of Cohen-Macaulay rings to the analysis of correctness of the greedy algorithm.

6.1 The algebraic framework

We will now go through the work in [Campillo & Revilla, 2001] for the necessary background on toric projective curves and their Cohen-Macaulay characterization.

Given a finite set of integers $M_h = \{1 = e_1 < e_2 < \ldots < e_h\}$, consider the subsemigroup S of $\mathbb{N}^2$ generated by $(0, e_h)$, $(e_1, e_h - e_1)$, $\ldots$, $(e_h, e_h - e_h)$, that is,

$$S = \langle (0, e_h), (e_j, e_h - e_j) : j = 1, 2, \ldots, h \rangle$$

and consider its projection on the second coordinate of $\mathbb{N}^2$:

$$S' = \langle e_h, e_h - e_1, e_h - e_2, \ldots, e_h - e_{h-1}, 0 \rangle$$

For each $i = 1, 2, \dots, e_h$, let c_i be the smallest integer in S' such that $c_i = e_h - i($ mod $e_h)$, i.e. $c_i = (e_h - i) + t \cdot e_h$ for some $t \geq 0$. For completeness, set $c_0 = 0$, and observe that $c_{e_i} = e_h - e_i$. Next, fix a field k and consider the semigroup algebra $k[S] := \bigoplus_{\alpha \in S} k\chi^\alpha$, with product given by $\chi^\alpha \cdot \chi^\beta = \chi^{\alpha+\beta}$ for $\alpha, \beta \in S$. The $\mathbb{N}$-grading on $k[S]$ given by $\deg(\chi^\alpha) = |\alpha|$, where $|\alpha| = b + c$ if $\alpha = (b, c)$, makes $k[S]$ the homogeneous coordinate algebra of a toric projective curve C_h. The curve C_h is *arithmetically Cohen-Macaulay* if the following property holds:

$$\boxed{\text{If } \alpha \in \mathbb{Z}^2,\ \alpha + (e_h, 0) \in S,\ \alpha + (0, e_h) \in S, \text{ then } \alpha \in S.} \tag{5}$$

Within this algebraic framework, Campillo and Revilla [Campillo & Revilla, 2001] have shown the following results that lead to a nice characterization of the correctness of the greedy strategy for the Coin-Exchange problem.

Proposition 3. *The toric projective curve C_h is arithmetically Cohen-Macaulay if and only if for all $i : 0 \leq i < e_h$, $(i, c_i) \in S$, where c_i is as defined above.* $\square$

Proposition 4. *C_h is arithmetically Cohen-Macaulay if and only if for all $B \geq e_h$, $opt(B) = opt(B - e_h) + 1$.* $\square$

Theorem 8. *For each $j \leq h$, set $M_j = \{1 = e_1, \dots, e_j\}$ and let C_j be the corresponding toric projective curve.*

(i) If the greedy algorithm correctly solves the Coin-Exchange problem for the coin system M_j, then C_j is arithmetically Cohen-Macaulay.
(ii) If $C_1, C_2, \dots, C_j$ are arithmetically Cohen-Macaulay, then the greedy algorithm correctly solves the Coin-Exchange problem for M_j. $\square$

Additionally the following characterization of $\text{opt}(B)$ in terms of the subsemigroup $S = \langle (0, e_h), (e_j, e_h - e_j) : j = 1, 2, \dots, h \rangle$ will be useful (this was originally established in [Campillo & Gimenez, 2000]).

Proposition 5. *For all $B \geq 0$, if c is the least integer such that $(B, c) \in S$ then $opt(B) = (B + c)/e_h$.* $\square$

6.2 *Finding an optimal solution for Coin-Exchange effectively within the algebraic framework*

Theorem 8 give us a way of showing the correctness of the greedy algorithm for a given coin system M_h: one has to check that all the local toric projective curves C_3, C_4, …, C_h are Cohen-Macaulay (where each curve C_j is associated to the system $M_j = \{e_1, \dots, e_j\}$). However, checking that each curve C_j is Cohen-Macaulay goes through checking membership in the associated semigroup S (Proposition 3), and this latter check depends on the universal property (5) to hold. So this strategy seems hard, in principle.

To find an effective way for checking that the greedy algorithm outputs an optimal solution for any given coin system, the idea is to actually check for those values where the greedy algorithm does not give optimal solutions, for it happens that *the set of witness where greedy fails for a given coin system is finite*. This remarkable and useful result was obtained by Kozen and Zaks in [Kozen & Zaks, 1994] through combinatorial arguments, but we shall derive it using the algebraic tools of Campillo and Revilla laid out in the previous section.

Theorem 9 ([Kozen & Zaks, 1994]). *Given a system of coins $M_h = \{1 = e_1 < e_2 < \ldots < e_h\}$. If for some B, greed$(B) >$ opt(B), then the smallest such B satisfies*

$$e_3 + 1 < B < e_h + e_{h-1}.$$

Proof. If $B < e_3$, we only need the subset $\{1, e_2\}$ of the coin system and for this $\mathrm{greed}(B) = \mathrm{opt}(B)$. For $B = e_3, e_3 + 1$, $\mathrm{greed}(B) = \mathrm{opt}(B) = 1, 2$.

For the other bound, let $B \geq e_h + e_{h-1}$, and assume that for all $x < B$, $\mathrm{greed}(x) = \mathrm{opt}(x)$. Let $i < e_h$ such that $B = te_h + i$, for some integer t (i.e. $B \equiv i \mod e_h$). Let c_i be the least integer in S' such that $c_i \equiv e_h - i \mod e_h$. Then $(i, c_i) \in S$ and $\mathrm{opt}(i) = (i + c_i)/e_h$. Also $(B, c_i) \in S$ and

$$\mathrm{opt}(B) \cdot e_h = B + c_i = te_h + i + c_i,$$

hence

$$\mathrm{opt}(B) = t + \frac{i + c_i}{e_h} = t + \mathrm{opt}(i)$$

Now, by assumption and the fact that $i < e_h$ we conclude that

$$\mathrm{opt}(B) = t + \mathrm{opt}(i) = t + \mathrm{greed}(i) = \mathrm{greed}(te_h + i) = \mathrm{greed}(B). \quad \square$$

Next, in order to avoid computing $\mathrm{opt}(B)$, Kozen and Zaks use the previous theorem to characterize greedy optimally solely in terms of $\mathrm{greed}(x)$. Again we give a proof of this fact using the algebraic setting for the Coin-Exchange problem:

Corollary 1 ([Kozen & Zaks, 1994]). *Given a system of coins $M_h = \{1 = e_1 < e_2 < \ldots < e_h\}$, the greedy algorithm is correct for M_h if and only if $\forall B \in (e_3 + 1, e_h + e_{h-1}), \forall c \in \{e_3, \ldots, e_h\}$ $(c < B \longrightarrow \mathit{greed}(B) \leq \mathit{greed}(B - c) + 1)$.*

Proof. $(\Rightarrow)$ Consider $B \in (e_3 + 1, e_h + e_{h-1})$ and e_j, for $j \in \{3, \ldots, h\}$, such that $e_j < B$. By hypothesis, Theorem 8 (i), Proposition 4 and Equation (4),

$$\mathrm{greed}(B) = \mathrm{opt}(B) = \mathrm{opt}(B - e_j) + 1 \leq \mathrm{greed}(B - e_j) + 1$$

$(\Leftarrow)$ If there is a B such that $\mathrm{greed}(B) > \mathrm{opt}(B)$, by Theorem 9 the smallest such B lies in $(e_3 + 1, e_h + e_{h-1})$. Let e_j be a coin used in the minimal representation of B. Then

$$\mathrm{opt}(B - e_j) = \mathrm{opt}(B) - 1 < \mathrm{greed}(B) - 1 \leq \mathrm{greed}(B - e_j)$$

contradicting that B is smallest witness of the failure of the greedy algorithm. $\square$

Using Corollary 1 one can effectively check that for the following coin systems the greedy algorithm always output optimal solutions:

1. US coin system.
2. Eurozone coin system.
3. Fibonacci coin system $\{1,2,3,5,8,13,21,44\}$.

On the other hand, the toric projective curves associated to each of these systems are examples of Cohen-Macaulay curves.

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