Bounded Queries to Arbitrary Sets

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Abstract

We prove that if $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$ for some k and an arbitrary set A, then A is reducible to its complement under a relativized nondeterministic conjunctive reduction. This result shows the first known property of arbitrary sets satisfying this condition, and implies some known facts such as Kadin's theorem [12] and its extension to the class $\mathrm{C_{=}P}$ [4, 7].

1 Introduction

We are interested in the hierarchy of sets accepted by machines that make a bounded number of queries to an arbitrary fixed set. Normally, only sets from some well-known complexity class, such as NP, have been considered as oracles for such hierarchies, and the results derived have widen our knowledge about the relationships among these classes or the hierarchies related to them (such as the boolean or the polynomial-time hierarchies). Here we consider arbitrary oracles in an attempt to generalize known results and provide a basis for further developments.

For any set A, we call $\mathrm{P}^{A[k]}$ the class of sets computable by deterministic polynomial-time machines that make k queries to oracle A. If we require the queries to be made in parallel, we denote the resulting class by $\mathrm{P}^{A\|[k]}$. Note that in order to decide sets in $\mathrm{P}^{A[k]}$, a machine can make a query to A that depends on the answers to previous queries (they are called serial or adaptive queries) while in the case of $\mathrm{P}^{A\|[k]}$ the queries depend exclusively on the input (parallel or non-adaptive queries).

The hierarchy

$$\mathrm{P}^{A[1]}, \mathrm{P}^{A[2]}, \mathrm{P}^{A[3]}, \ldots$$

is called the bounded-query hierarchy relative to A, while its parallel counterpart is called the bounded parallel query hierarchy relative to A [3]. Sometimes we will consider the bounded query hierarchies relative to classes, instead of sets; they are defined in the

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obvious way: for a class of sets $\mathcal{C}$, $\mathrm{P}^{\mathcal{C}[k]}$ is $\bigcup_{A\in\mathcal{C}} \mathrm{P}^{A[k]}$. We will use the property that if A is a $\le_m^{\mathrm{P}}$-complete set for $\mathcal{C}$, then $\mathrm{P}^{\mathcal{C}[k]} = \mathrm{P}^{A[k]}$.

Some properties about sets whose bounded query hierarchies collapse are known when these sets are taken from a uniform class. The results of Kadin [12] ($\mathrm{P}^{\mathrm{NP}[k]} = \mathrm{P}^{\mathrm{NP}[k+1]}$ implies $\mathrm{PH} = \Delta_3^p$), Chang [6] ($A \in \mathrm{NP}$ and $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$ imply $A \in \widehat{\mathrm{low}_3}$), Beigel, Chang, and Ogiwara [4], and Green [7] ($\mathrm{P}^{\mathrm{C_=P}[k]} = \mathrm{P}^{\mathrm{C_=P}[k+1]}$ implies $\mathrm{PH}^{\mathrm{PP}} = \mathrm{P}^{\mathrm{NP}^{\mathrm{PP}}}$) can all be seen as consequences of the collapse of different bounded query hierarchies. However, no properties of an arbitrary set A satisfying the simple equation $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$, for some k, were known. We prove in this paper that, in this case, A is reducible to its complement under a weak reducibility.

In order to state the main result more precisely, consider the following nondeterministic reducibilities, defined in [13]. We say that a set A is $\le_m^{\mathrm{NP}}$-reducible to a set B if there is a nondeterministic polynomial-time machine that, on input x, generates a word belonging to B, for some computation path, if and only if $x \in A$. Similarly, we say that A is $\le_{ctt}^{\mathrm{NP}}$-reducible to B if there is a nondeterministic polynomial-time machine that, on input x, generates a tuple of words that belong to B, for some computation path, if and only if $x \in A$. For any reducibility $\le_r^{\mathrm{NP}}$, we write $\le_r^{\mathrm{NP},C}$ if the machine defining the reducibility has unrestricted access to oracle C.

In Section 3 we prove that if the bounded-query hierarchy relative to A collapses, then there exists a sparse set S such that $A \le_{ctt}^{\mathrm{NP},S} \overline{A}$. In Section 4 we show that the previously known consequences of the collapse of some bounded query hierarchies can be proved from our general theorem, in particular Kadin's theorem [12] and a similar result for the class $\mathrm{C_=P}$ [4, 7].

2 Preliminaries

We assume that the reader is familiar with the basic structural complexity notions, and in particular with classes such as P, NP, and PH. The main result in this paper will be applied to the classes NP and $\mathrm{C_=P}$. Let $\Sigma = \{0,1\}$, and define the class $\mathrm{C_=P}$ as follows:

Definition 2.1 *[17, 19] A set A is in $\mathrm{C_=P}$ if there exists a polynomial time nondeterministic Turing machine M and a polynomial time computable function f such that for every $x \in \Sigma^*$, it holds that $x \in A$ if and only if the number of accepting computation paths of M on input x is equal to $f(x)$.*

We denote with co-$\mathrm{C_=P}$ (or, alternatively, $\mathrm{C_{\neq}P}$) the class of sets whose complements belong to $\mathrm{C_=P}$. Also, the notation NE stands for the class $\bigcup_c \mathrm{NTIME}[2^{cn}]$, and NEXP for $\bigcup_k \mathrm{NTIME}[2^{n^k}]$. Following [6], we construct "boolean languages" with respect to a set.

Definition 2.2 *For any set A, we define the following sets:*

$$\begin{aligned}
\mathrm{BL}_A(1) &= A \\
\mathrm{BL}_A(2k) &= \{(x_1,\ldots,x_{2k-1},x_{2k}) : (x_1,\ldots,x_{2k-1}) \in \mathrm{BL}_A(2k-1) \textit{ and } x_{2k} \in \overline{A}\} \\
\mathrm{BL}_A(2k+1) &= \{(x_1,\ldots,x_{2k},x_{2k+1}) : (x_1,\ldots,x_{2k}) \in \mathrm{BL}_A(2k) \textit{ or } x_{2k+1} \in A\}
\end{aligned}$$

$$\begin{aligned} \text{co-BL}_A(1) &= \overline{A} \\ \text{co-BL}_A(2k) &= \{(x_1,\ldots,x_{2k-1},x_{2k}) : (x_1,\ldots,x_{2k-1}) \in \text{co-BL}_A(2k-1) \ \textit{or}\ x_{2k} \in A\} \\ \text{co-BL}_A(2k+1) &= \{(x_1,\ldots,x_{2k},x_{2k+1}) : (x_1,\ldots,x_{2k}) \in \text{co-BL}_A(2k) \ \textit{and}\ x_{2k+1} \in \overline{A}\} \end{aligned}$$

We say that a set S is *self-p-printable* if there is a polynomial-time oracle Turing machine which prints all the strings in S of a given length n on input 1^n. Other sets that are used in the proofs are defined next. Note that the first two sets are defined with different notations; this will help to make easier the proof of Claim 2, in next section.

Definition 2.3 *For any set A we define the following sets:*

$$\begin{aligned} A^{\forall} &= \{x_1\#x_2\#\cdots\#x_n : n \geq 1 \ \textit{and}\ (\forall i \leq n)[x_i \in A]\} \\ A^{\exists\forall} &= \{(x_1,\ldots,x_n) : n \geq 1 \ \textit{and}\ (\exists i \leq n)[x_i \in A^{\forall}]\} \\ \text{PARITY}_k^A &= \{(x_1,\ldots,x_k) : \|\{i : 1 \leq i \leq k \ \textit{and}\ x_i \in A\}\| \ \textit{is odd}\ \} \end{aligned}$$

We define now the relativized nondeterministic reducibilities $\leq_m^{\mathrm{NP}}$ and $\leq_{ctt}^{\mathrm{NP}}$ (see [13, 1] for the nonrelativized versions).

Definition 2.4 *For any three sets A, B and C, we say that*

(1) A is polynomial-time nondeterministic many-one reducible to B relative to C (denoted $A \leq_m^{\mathrm{NP},C} B$), if there exists a polynomial-time nondeterministic oracle Turing transducer M such that for every $x \in \Sigma^*$,

(a) for each computation path of M on input x and oracle C, M outputs some string, and

(b) $x \in A$ if and only if there exists some computation path of M on input x and oracle C that outputs some string $y \in B$.

(2) A is polynomial-time nondeterministic conjunctive truth-table reducible to B relative to C (denoted $A \leq_{ctt}^{\mathrm{NP},C} B$), if there exists a polynomial-time nondeterministic oracle Turing transducer M such that for every $x \in \Sigma^*$,

(a) for each computation path of M on input x and oracle C, M outputs a string of the form $y_1\#\ldots\#y_k$, and

(b) $x \in A$ if and only if there exists some computation path of M on input x and oracle C that outputs some string $y_1\#\ldots\#y_k$ such that $\{y_1,\ldots,y_k\} \subseteq B$.

Definition 2.5 *For any set B, we define the following classes related to the above nondeterministic reducibilities:*

$$\begin{aligned} \mathrm{NP}_m^B &= \{A : A \leq_m^{\mathrm{NP}} B\} \\ \mathrm{NP}_{ctt}^B &= \{A : A \leq_{ctt}^{\mathrm{NP}} B\} \end{aligned}$$

Let us mention two useful observations about hierarchy collapses. The first one indicates that the collapse to a certain level propagates to the infinitely many superior levels, and is called *upward collapse property* in [11].

Observation 2.6 *1. If* $\mathrm{P}^{A\|[k]} = \mathrm{P}^{A\|[k+1]}$*, then for all* $j > k$*,* $\mathrm{P}^{A\|[k]} = \mathrm{P}^{A\|[j]}$.

2. [11] If $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$*, then for all* $j > k$*,* $\mathrm{P}^{A[k]} = \mathrm{P}^{A[j]}$.

Also note that there is a tight relation between the bounded-query hierarchy and the bounded parallel query hierarchy. This was already noted in [3] for bounded query hierarchies relative to arbitrary classes of sets.

Observation 2.7 *For any set* A*,* $(\exists i)[\mathrm{P}^{A[i]} = \mathrm{P}^{A[i+1]}] \iff (\exists j)[\mathrm{P}^{A\|[j]} = \mathrm{P}^{A\|[j+1]}]$.

3 The Main Result

In this section we show that if for some set A, there is a k such that $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$, then A is reducible to its complement under a weak reduction: $A \leq_{ctt}^{\mathrm{NP},S} \overline{A}$, for a certain sparse set S. After this, we will see how Kadin's theorem and its extension to the class $\mathrm{C_{=}P}$ can be easily derived from our result.

The next result derives directly from a technical lemma of Chang [6].

Lemma 3.1 *If* $\mathrm{BL}_A(k) \leq_m^{\mathrm{P}}$ co-$\mathrm{BL}_B(k)$ *for some* k*, then* $\overline{A} \leq_m^{\mathrm{NP},S} B$*, where* S *is a self-p-printable set in* $\mathrm{P}^{\mathrm{NP}^{\mathrm{NP}_m^B}}$.

Proof The same proof than that of Lemma 1 in [6] can be used to prove this lemma. The difference is that we take two arbitrary sets A and B (not necessarily from NP as in [6]), and then the final NP^S algorithm that decides $\overline{A}$ must be considered in this case as the machine that witnesses the reduction $\overline{A} \leq_m^{\mathrm{NP},S} B$. Also, the first algorithm in that proof shows that $S \in \Delta_3^p$ but, as B is here an arbitrary set, it is not hard to check that $S \in \mathrm{P}^{\mathrm{NP}^{\mathrm{NP}_m^B}}$. □

Now, we can prove the main theorem.

Theorem 3.2 *For any set* A*, if* $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$ *for some* k*, then* $A \leq_{ctt}^{\mathrm{NP},S} \overline{A}$*, where* S *is a self-p-printable set in* $\mathrm{P}^{\mathrm{NP}^{\mathrm{NP}_{ctt}^{\overline{A}}}}$.

Proof The hypothesis implies, using the observations of the previous section, that $\mathrm{P}^{A\|[2^k+1]} = \mathrm{P}^{A\|[k]}$ for some k, and then the set $\mathrm{BL}_{\overline{A}}(2^k+1)$ is in $\mathrm{P}^{A\|[k]}$.

We claim that the set $\mathrm{PARITY}_{2^k}^{A^{\forall}}$ is $\leq_m^{\mathrm{P}}$-hard for $\mathrm{P}^{A\|[k]}$ and that for any odd n, $\mathrm{PARITY}_n^{A^{\forall}} \leq_m^{\mathrm{P}}$ co-$\mathrm{BL}_{\overline{A^{\exists\forall}}}(n)$. These two claims allow us to prove the following chain of reductions:

$$\begin{aligned}
\mathrm{BL}_{\overline{A}}(2^k+1) &\leq_m^{\mathrm{P}} \mathrm{PARITY}_{2^k}^{A^{\forall}} && \text{by Claim 1} \\
&\leq_m^{\mathrm{P}} \mathrm{PARITY}_{2^k+1}^{A^{\forall}} \\
&\leq_m^{\mathrm{P}} \text{co-}\mathrm{BL}_{\overline{A^{\exists\forall}}}(2^k+1) && \text{by Claim 2}
\end{aligned}$$

Now, Lemma 3.1 implies that $A \leq_m^{\mathrm{NP},S} \overline{A^{\exists\forall}}$, where S is a self-p-printable set in $\mathrm{P}^{\mathrm{NP}^{\mathrm{NP}_m^{\overline{A^{\exists\forall}}}}}$. But we have to express this in terms of $\overline{A}$ instead of $\overline{A^{\exists\forall}}$ in order to prove the theorem.

To do this, we only need to show that a nondeterministic many-one reduction to $\overline{A^{\exists\forall}}$ can be transformed into a nondeterministic conjunctive truth-table reduction to $\overline{A}$ (both of them relativized or not). For example, in the case of nonrelativized reducibilities, assume that $L \leq_m^{\mathrm{NP}} \overline{A^{\exists\forall}}$ and that it is witnessed by the nondeterministic Turing machine M. We can define another nondeterministic Turing machine N that works as follows. On any input, it starts simulating M and if on a certain computation path M outputs the string w, N computes it but does not output anything yet. If w does not have the format of the strings in $A^{\exists\forall}$ then it is in $\overline{A^{\exists\forall}}$ and M accepts, so N writes some fixed string belonging to $\overline{A}$ in order to accept. Otherwise, assume that w has the format of the strings in $A^{\exists\forall}$: $w = (y_1, \ldots, y_l)$. If $w \in \overline{A^{\exists\forall}}$, each subtuple y_i must contain a string in $\overline{A}$. Thus, the machine N only has to guess a string of each subtuple that is a candidate to be in $\overline{A}$, and output this sequence of strings for this computation path. This machine shows that $L \leq_{ctt}^{\mathrm{NP}} \overline{A}$.

The above idea can be applied to the reduction $A \leq_m^{\mathrm{NP},S} \overline{A^{\exists\forall}}$, showing that $A \leq_{ctt}^{\mathrm{NP},S} \overline{A}$, as well as to the class $\mathrm{NP}_m^{\overline{A^{\exists\forall}}}$, showing that it is contained in $\mathrm{NP}_{ctt}^{\overline{A}}$. This proves the theorem.

Next, we prove the claims. The first one contains some arguments used in the proof of Theorem 2.2 of Chapter 1 in [20]. We reproduce them here for completeness.

Claim 1 *For any k, $\mathrm{PARITY}_{2^k}^{A^\forall}$ is $\leq_m^{\mathrm{P}}$-hard for $\mathrm{P}^{A\|[k]}$.*

Proof Let $L \in \mathrm{P}^{A\|[k]}$, and let M be a machine that witnesses it. Let $q_i(x)$ be the i-th query made by M on input x.

Given an instance x, it is easy to compute a boolean formula $f_x(x_1, \ldots, x_k)$ in polynomial time, such that:

$$x \in L \iff f_x(\chi_A(q_1(x)), \ldots, \chi_A(q_k(x))) = 1 \tag{1}$$

We can test whether M accepts x on every possible combination of answers to the k queries, and so write f_x in disjunctive normal form. Next, we compute the ring sum expansion of this formula in order to make it similar to the form of the instances of the set $\mathrm{PARITY}_{2^k}^{A^\forall}$. Every term in f_x, when written in disjunctive normal form, corresponds to a different vector of answers with which M accepts and, hence, at most one of the terms in f_x can be true. This means that the disjunctions are actually exclusive, and we can replace them by $\oplus$-operators. We also replace every negation $\overline{z}$ by $z \oplus 1$. At this moment, f_x is expressed in terms of the operators $\wedge$ and $\oplus$, so we may get a $\oplus$-sum of $\wedge$-products by applying the distributivity law. If a term $\bigwedge_{i \in X} x_i$ appears an odd number of times then we set the constant $a_X = 1$. Thus, f_x is equivalent to the formula:

$$\bigoplus_{X \subseteq \{1, \ldots, k\}} a_X \wedge \bigwedge_{i \in X} x_i$$

Now we can define the reduction function from L to $\mathrm{PARITY}_{2^k}^{A^\forall}$. Fix a string $s \notin A$, and call X_i the i-th subset of the set $\{1, \ldots, k\}$ in some specific ordering. Given x, we construct a tuple $l = (t_1, \cdots, t_{2^k})$ which consists of 2^k instances of $A^\forall$ such that for all $i \leq 2^k$:

- if $a_{X_i} = 0$ then t_i contains the string s
- if $a_{X_i} = 1$ then t_i contains the strings $q_j(x)$ such that $j \in X_i$ $(1 \leq j \leq k)$

Obviously, formula f_x is true on inputs $\chi_A(q_1(x)), \ldots, \chi_A(q_k(x))$ if and only if its ring sum expansion evaluates to true on the same inputs, and this happens when there is an odd number of terms $a_X \wedge \bigwedge_{i \in X} x_i$ which are true. But note that for any $i \leq 2^k$, $a_X \wedge \bigwedge_{i \in X} x_i$ is true if and only if $t_i \in A^\forall$, according to the definition of the tuple l. It is not hard to see now that

$$f_x(\chi_A(q_1(x)), \ldots, \chi_A(q_k(x))) = 1 \Longleftrightarrow l \in \mathrm{PARITY}_{2^k}^{A^\forall} \tag{2}$$

So, from equivalences 1 and 2 and the fact that l can be computed in time polynomial in $|x|$, we conclude that this constitutes a polynomial-time reduction from an arbitrary set L in $\mathrm{P}^{A\|[k]}$ to $\mathrm{PARITY}_{2^k}^{A^\forall}$. □

Claim 2 *For any odd n,* $\mathrm{PARITY}_n^{A^\forall} \leq_m^{\mathrm{P}} \text{co-BL}_{\overline{A^{\exists\forall}}}(n)$.

Proof Let x be an instance of $\mathrm{PARITY}_n^{A^\forall}$. Then, it is of the form $x = (x_1, \ldots, x_n)$, where for any $i \leq n$, x_i is an instance of $A^\forall$.

Define the reduction function $f(x) = (f_n(x), \ldots, f_1(x))$, where:

$$\begin{aligned}
f_1(x) &= (x_1, \ldots, x_n) = x \\
f_2(x) &= (x_1\#x_2,\ x_1\#x_3, \ldots, x_2\#x_3, \ldots, x_{n-1}\#x_n) \\
&\vdots \\
f_i(x) &= (x_1\#\cdots\#x_{i-1}\#x_i,\ x_1\#\cdots\#x_{i-1}\#x_{i+1}, \ldots, x_{n-i+1}\#\cdots\#x_n) \\
&\vdots \\
f_n(x) &= (x_1\#x_2\#\cdots\#x_n)
\end{aligned}$$

For any $i \leq n$, $f_i(x)$ contains the $\binom{n}{i}$ combinations of i elements of the input $x = (x_1, \ldots, x_n)$. There is at least one such combination, a component of $f_i(x)$, belonging to $A^\forall$ if and only if there are at least i elements x_j's in $A^\forall$ (note that here we use that $A^\forall$ is closed under concatenation), and this happens if and only if $f_i(x) \in A^{\exists\forall}$. Then, the fact $f_i(x) \in A^{\exists\forall}$ means that there are at least i components of the input x in $A^\forall$; so, with an adequate combination of the facts $f_1(x) \in A^{\exists\forall}$, $f_2(x) \in A^{\exists\forall}$, …, $f_n(x) \in A^{\exists\forall}$, we can decide whether there is an odd number of components of x in $A^\forall$. That is to say, $x \in \mathrm{PARITY}_n^{A^\forall}$, if and only if

$$f_1(x) \in A^{\exists\forall} \wedge (f_2(x) \in \overline{A^{\exists\forall}} \vee (f_3(x) \in A^{\exists\forall} \wedge \cdots$$

and this holds, by definition, if and only if $f(x) \in \text{co-BL}_{\overline{A^{\exists\forall}}}(n)$. □

□ **Proof of theorem 3.2**

4 Some Consequences: the Bounded Query Hierarchies over NP and $\mathrm{C_{=}P}$

The most interesting applications of Theorem 3.2 can be found for classes which are closed under $\leq_{ctt}^{\mathrm{NP}}$-reducibility, property satisfied by any class that is closed under the $\leq_{m}^{\mathrm{NP}}$ and $\leq_{ctt}^{\mathrm{P}}$-reducibilities. Three classes that satisfy these conditions are NP, $\mathrm{C_{\neq}P}$ and NEXP [1]. We show now the consequences for the first two classes (for NE, read the comment in the next section).

Now, we show that if we take the set A in our main result from the class of NP-complete sets (for $\leq_{m}^{\mathrm{P}}$-reducibility), we obtain Kadin's theorem.

Corollary 4.1 *[12] If* $\mathrm{P}^{\mathrm{NP}[k]} = \mathrm{P}^{\mathrm{NP}[k+1]}$ *for some* k*, then* $\mathrm{PH} \subseteq \Delta_3^p$.

Proof The hypothesis is equivalent to the equality $\mathrm{P}^{\mathrm{SAT}[k]} = \mathrm{P}^{\mathrm{SAT}[k+1]}$, which is also equivalent to $\mathrm{P}^{\overline{\mathrm{SAT}}[k]} = \mathrm{P}^{\overline{\mathrm{SAT}}[k+1]}$. Theorem 3.2 implies that $\overline{\mathrm{SAT}} \leq_{ctt}^{\mathrm{NP},S} \mathrm{SAT}$ for some self-p-printable set S in $\mathrm{P}^{\mathrm{NP}^{\mathrm{NP}_{ctt}^{\mathrm{SAT}}}}$. As NP is closed under $\leq_{ctt}^{\mathrm{NP}}$-reducibility, this means that co-NP $\subseteq \mathrm{NP}^{S}$ for some self-p-printable set S in Δ_3^p, which implies that $\mathrm{PH} \subseteq \Delta_3^p$, by the results in [10]. □

The next result was proved by Green [7] and independently, in a stronger form, by Beigel, Chang, and Ogiwara [4]. We present it here also as a consequence of our main result.

Corollary 4.2 *[7, 4] If* $\mathrm{P}^{\mathrm{C_{=}P}[k]} = \mathrm{P}^{\mathrm{C_{=}P}[k+1]}$ *for some* k*, then* $\mathrm{PH}^{\mathrm{PP}} \subseteq \mathrm{P}^{\mathrm{NP}^{\mathrm{PP}}}$.

Proof By a similar argument to that of the previous corollary, we can state that $\mathrm{C_{=}P} \subseteq \mathrm{C_{\neq}P}^{S}$, for some self-p-printable set S in $\mathrm{P}^{\mathrm{NP}^{\mathrm{C_{\neq}P}}}$, since $\mathrm{C_{\neq}P}$ is closed under $\leq_{ctt}^{\mathrm{NP}}$-reducibility [1].

Consider the inclusions $\mathrm{NP}^{\mathrm{C_{\neq}P}^{S}} \subseteq \exists \mathrm{C_{\neq}P}^{S} \subseteq \mathrm{C_{\neq}P}^{S}$, where the first one derives from Torán [18], and the second one from the fact that $\mathrm{C_{\neq}P}$ is closed under $\leq_{m}^{\mathrm{NP}}$-reducibility. By induction we have that $\mathrm{PH}^{\mathrm{C_{\neq}P}^{S}} \subseteq \mathrm{C_{\neq}P}^{S}$.

Now, also from [18] we know that $\mathrm{PH}^{\mathrm{PP}} \subseteq \mathrm{PH}^{\mathrm{C_{=}P}}$, and then $\mathrm{PH}^{\mathrm{PP}} \subseteq \mathrm{PH}^{\mathrm{C_{\neq}P}^{S}} \subseteq \mathrm{C_{\neq}P}^{S}$. But this last class can be shown to be included in $\mathrm{P}^{\mathrm{NP}^{\mathrm{C_{\neq}P}}}$, by using the information that we have about S, and hence in $\mathrm{P}^{\mathrm{NP}^{\mathrm{PP}}}$. Therefore $\mathrm{PH}^{\mathrm{PP}} \subseteq \mathrm{P}^{\mathrm{NP}^{\mathrm{PP}}}$. □

5 Conclusions and Open Problems

We have given the first known property for an arbitrary set A when it satisfies $\mathrm{P}^{A[k]} = \mathrm{P}^{A[k+1]}$ for some k. We have proved that, in this case, A is reducible to its complement under a weak reduction: $A \leq_{ctt}^{\mathrm{NP},S} \overline{A}$, for some sparse set S (whose complexity is specified in terms of that of A). We have also seen that this leads us to conclude some known facts

[1] The case of NP is well known. The closure of $\mathrm{C_{\neq}P}$ under $\leq_{m}^{\mathrm{NP}}$-reducibility is proved in [7] and both this and the closure under $\leq_{ctt}^{\mathrm{P}}$-reducibility can be derived from [4, 8, 14, 16]. For NE, the proof is similar to that of NP. For related results, see [15].

about the collapse of the bounded query hierarchies relative to NP and $C_=P$ (or about the collapse of their respective boolean hierarchies, which is equivalent in these cases). This means that our result is a good generalization of the mentioned results for the classes NP and $C_=P$.

A first question left open in this paper is: Can we conclude some interesting consequence for some other complexity class? It seems interesting to apply our theorem to the class of nondeterministic exponential time since, although Hemachandra [9] proved the collapse of the strong exponential hierarchy to P^{NE}, it is not known whether a stronger collapse (as $P^{NE[k]} = P^{NE[k+1]}$) could cause some unlikely consequence. In fact, the main result in this paper implies that if the bounded-query hierarchy over NE collapses, then co-NEXP $\subseteq$ NEXP/poly [2]. Does this imply that NEXP = co-NEXP? Buhrman and Homer study in [5] some questions similar to this one.

Apparently more difficult problems arise when one tries to obtain some consequence from the equality of the classes $P^{A[log]}$ and P^A. Note that, however, we know some consequences of the facts [3] $PF^{A[log]} = PF^A$, $PF^{A[k]} = PF^{A[k+1]}$, and now $P^{A[k]} = P^{A[k+1]}$. This seems to be a hard question since, by taking $A = SAT$, this would give us a consequence of the fact $\Delta_2^p = \Theta_2^p$, which is still unknown.

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[2] Note that it is easy to prove by padding arguments that the bounded-query hierarchy over NE collapses if and only if the bounded-query hierarchy over NEXP collapses.

[3] For the first one, from the results by Amir, Beigel, and Gasarch [2] it follows that $PF^{A[log]} = PF^A$ implies that $A \in$ NP/poly $\cap$ co-NP/poly. The second fact is equivalent to saying that A is cheatable; also in [2] it is shown that in this case $A \in$ P/poly.

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