Paramodulation with Non-Monotonic Orderings*

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Abstract

All current completeness results for ordered paramodulation require the term ordering $\succ$ to be well-founded, monotonic and total(izable) on ground terms. Here we introduce a new proof technique where the only properties required for $\succ$ are well-foundedness and the subterm property[1]. The technique is a relatively simple and elegant application of some fundamental results on the termination and confluence of ground term rewrite systems (TRS).

By a careful further analysis of our technique, we obtain the first Knuth-Bendix completion procedure that finds a convergent TRS for a given set of equations E and a (possibly non-totalizable) reduction ordering $\succ$ whenever it exists[2]. Note that being a reduction ordering is the minimal possible requirement on $\succ$, since a TRS terminates if, and only if, it is contained in a reduction ordering.

Keywords: *term rewriting, automated deduction.*

1 Introduction

Deduction with equality is fundamental in mathematics, logics and many applications of formal methods in computer science. During the last two decades this field has importantly progressed through new Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses. These techniques have lead to important results on theorem proving in first-order logic with equality [11, 2, 1, 3] (that have been applied to state-of-the-art theorem provers like Spass [16]), results on logic-based complexiy and decidability analysis [5, 13], on deduction with *constrained clauses* [12, 14], and on many other applications like inductive theorem proving, symbolic constraint solving, or equational-logic programming.

But all current completeness results for Knuth-Bendix completion and ordered paramodulation require the term ordering $\succ$ to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms. All main proof techniques, like the transfinite semantic tree method [11], the proof ordering method [2, 1], and the model generation method [3] rely at some point on these requirements. Moreover, in many practical situations these requirements are too strong.

For example, one may want to apply completion to a terminating set of rules with a given orientation by a reduction ordering that cannot be extended to a total one, like $f(a) \to f(b)$ and $g(b) \to g(a)$, for which a and b must be uncomparable in any monotonic extension.

Another typical situation is deduction modulo built-in equational theories E, where the existence of a total E-*compatible* reduction ordering is a very strong requirement. For example, the existence of such an ordering for the case where E consists of associativity and commutativity (AC) properties for some symbols remained open for a long time, and, once it was found, it triggered quite a number of results, like the decidability of the ground AC-word and -unification problems. Unfortunately, for many E such orderings cannot exist. For instance, when E contains an idempotency axiom $f(x, x) = x$, then if $s \succ t$, by monotonicity one should have $f(s, s) \succ f(s, t)$, which by E-compatibility implies $s \succ f(s, t)$ and hence non-well-foudedness.

In this paper we introduce techniques for dropping the monotonicity requirement that open the door to deduction modulo many more classes of equational theories. The only properties required for $\succ$ are well-foundedness and the subterm property. Our technique (Sections 3 to 6) is a variant of the model generation technique, with the main difference that the termination of the ground rewrite system R that defines the model is not a consequence of the ordering, and hence termination is proved otherwise. Induction on (an extension of) the reduction ordering induced by R is then used

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[1] This solves a well-known open problem (e.g. at the RTA list of open problems since 1995).

[2] This was an even better known open problem (e.g. posed by N. Dershowitz and J-P. Jouannaud on the RTA list of open problems since its creation in 1991).

for proving the main completeness theorem. Our results are given here for paramodulation with general first-order clauses with *selection* of negative literals.

By a careful further analysis of our technique, we also obtain the first practical Knuth-Bendix completion procedure (Section 7) that finds a convergent TRS for a given set of equations E and a (possibly non-totalizable) reduction ordering $\succ$ whenever it exists. Note that being a reduction ordering is the minimal possible requirement on $\succ$, since a TRS terminates if, and only if, it is contained in a reduction ordering.

Finally, in Section 8 we discuss some directions for future work (perhaps more new open problems are created than old ones are closed in this paper), giving counter examples indicating the limitations for some of the extensions.

2 Basic Notions

We use the standard definitions of [6]: $T(\mathcal{F},\mathcal{X})$ ($T(\mathcal{F})$) is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$), the subterm of t at *position* p is denoted $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$.

If $\to$ is a binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\to^+$ is its transitive closure and $\to^*$ is its reflexive-transitive closure. We write $s \to^! t$ if $s \to^* t$ and there is no t' such that $t \to t'$. Then t is called *irreducible* and a *normal form* of s (w.r.t. $\to$). The relation $\to$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \to s_2 \to \ldots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \to^*$ is contained in $\to^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \to \ \subseteq \ \to^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\to$ on terms is *monotonic* if $s \to t$ implies $u[s]_p \to u[t]_p$ for all terms s, t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset $\{s,t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$. A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \to \Delta$. The empty clause $\square$ is a clause where both Γ and Δ are empty.

A *rewrite rule* is an ordered pair of terms (s,t), written $s \to t$, and a set of rewrite rules R is a *term rewrite system* (TRS). The rewrite relation with R on $T(\mathcal{F},\mathcal{X})$, denoted $\to_R$, is the smallest monotonic relation such that $l\sigma \to_R r\sigma$ for all $l \to r \in R$ and all substitution σ, and if $s \to_R t$ then we say that *s rewrites into t* with R. R is called terminating, confluent, etc. if $\to_R$ is. A rewrite system R is *convergent* if it is confluent and terminating; then every term t has a unique normal form w.r.t. $\to_R$, denoted by $nf_R(t)$, and $s \simeq t$ is a logical consequence of R (where R is seen as a set of equations) iff $nf_R(s) = nf_R(t)$.

Let R be a set of ground equations or rewrite rules. Then the congruence $\leftrightarrow_R^*$ defines an equality Herbrand interpretation denoted by R^*, where the only predicate $\simeq$ is interpreted by $s \simeq t$ iff $s \leftrightarrow_R^* t$. We write $s = t \in R^*$ if $s \leftrightarrow_R^* t$. R^* satisfies (is a model of) a ground clause $\Gamma \to \Delta$, denoted $R^* \models \Gamma \to \Delta$, if $R^* \not\supseteq \Gamma$ or $R^* \cap \Delta \neq \emptyset$. The empty clause $\square$ is hence satisfied by no interpretation. R^* satisfies a set of clauses S, denoted by $R^* \models S$, if it satisfies every clause in S. For dealing with non-equality predicates, atoms A can be expressed by equations $A \simeq$ *true* where *true* is a new symbol.

3 Some properties of ground TRS and orderings

A (strict partial) *ordering* on $T(\mathcal{F},\mathcal{X})$ is an irreflexive transitive relation $\succ$. It is a *reduction* ordering if it is well-founded and monotonic, and stable under substitutions: $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

Definition 1 *A* west-ordering *is a well-founded ordering on $T(\mathcal{F})$ that fulfils the subterm property and that is total on $T(\mathcal{F})$ (it is called* west *after well-founded, subterm and total).*

Not all well-founded orderings on terms can be extended to west-orderings, even if they do not contradict the subterm property. For example, if $a \succ_1 f(b)$ and $b \succ_1 f(a)$, then, if $\succ$ is $(\succ_1 \cup \rhd)^+$, we get $a \succ f(b) \succ b \succ f(a) \succ a$. But every well-founded ordering can be totalized [15], and hence every well-founded ordering satisfying the subterm property can be extended to a west ordering. We also have the following:

Lemma 2 *Every reduction ordering $\succ_r$ can be extended to a west ordering.*

Proof: Let $\succ_{rs}$ be $(\succ_r \cup \rhd)^+$. Then $\succ_{rs}$ is well-founded. We will derive a contradiction from the existence of an infinite sequence $s_1 \succ_{rs} s_2 \succ_{rs} \ldots$ with s_1 minimal w.r.t. $\succ_r$. By monotonicity of $\succ_r$ we have that $s[t] \rhd t \succ_r u$ implies $s[t] \succ_r s[u] \rhd u$, i.e. both relations commute. Now since $\rhd$ is well-founded there should be some s_k in the sequence which is the first one such that $s_k \succ_r s_{k+1}$, and hence by applying the commutation property we can re-arranged the sequence obtaining an infinite sequence $s_1 \succ_r s'_2 \succ_{rs} \ldots s'_k \succ_{rs} s_{k+1} \succ_{rs} \ldots$ for some $s'_2,\ldots,s'_k$, which contaradicts the minimality of s_1. $\square$

Lemma 3 *Let R be a ground TRS such that for all rules $l \to r$ in R the term r is irreducible by R. Then R is terminating.*

Proof: Assume R is non-terminating. Then there exists an infinite rewrite sequence $t_1 \rightarrow_R t_2 \rightarrow_R \ldots$ It is easy to extract an infinite subsequence $s_1 \rightarrow_R s_2 \rightarrow_R \ldots$ of it where there is at least one rewrite step $s_i \rightarrow_R s_{i+1}$ at the topmost position, i.e., where $s_i \equiv l$ and $s_{i+1} \equiv r$ for some rule $l \rightarrow r$ in R. But then s_{i+1} is irreducible by R, contradicting the infiniteness assumption. □

Lemma 4 *Let $\succ$ be a west ordering, and let R be a ground TRS such that, for all $l \rightarrow r$ in R, $l \succ r$ and r is irreducible by R at non-topmost positions. Then R is terminating.*

Proof: If R is non-terminating there is an infinite sequence $s_1 \rightarrow_R s_2 \rightarrow_R \ldots$ with at least one rewrite step $s_i \rightarrow_R s_{i+1}$ at the topmost position, i.e., where $s_i \equiv l$ and $s_{i+1} \equiv r$ for some rule $l \rightarrow r$ in R. But then $s_{i+1} \rightarrow_R s_{i+2} \rightarrow_R \ldots$ is an infinite sequence with steps only at topmost positions, where $s_{i+1} \succ s_{i+2} \succ \ldots$ contradicting the well-foundedness of $\succ$. □

Finally we will also use the following well-known results on orderings:

Lemma 5 ([6]) *If R is a terminating TRS then $\rightarrow_R^+$ is a reduction ordering.*

Given an ordering $\succ$, we define $\succ_{mul}$ to be the smallest ordering on multisets such that: $M \cup \{s\} \succ_{mul} N \cup \{t_1, \ldots, t_n\}$ if $M = N$ and $s \succ t_i$ for all $i \in 1 \ldots n$. If $\succ$ is well founded and total on S, so is $\succ_{mul}$ on finite multisets over S [7].

4 Paramodulation with equations

We first introduce part of our ideas by explaining the purely equational case, whose results will be used in Section 7. In the following, let $\succ$ be a given west ordering.

Definition 6 *The inference rule of (equational)* ordered paramodulation *with respect to $\succ$ is:*

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$, the most general unifier of $s|_p$ and l, where $s|_p$ is not a variable, and l is maximal in its premise. That is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$.

The usefulness of the ordering restrictions in practical applications depends on the way the west ordering $\succ$ is given. Normally, what is known is an approximation given by a non-total ordering $\succ_v$ on terms with variables such that $s \succ_v t$ implies $s\sigma \succ t\sigma$ for all ground σ. This is the case in particular if $\succ_v$ is a reduction ordering of which $\succ$ is an extension, as in Lemma 2. Then the inference is not needed if, for instance, $r\sigma \succeq_v l\sigma$.

We now define, by induction on $\succ_{mul}$, a ground TRS R_E *generated* by a set of equations E:

Definition 7 *Let E be a set of equations. An instance e of the form $l \simeq r$ of an equation in E* generates *the rule $l \rightarrow r$ if*

1. *$l \succ r$, and*
2. *l and r are irreducible by R_e*

where R_e is the set of rules generated by all instances d of equations in E such that $e \succ_{mul} d$. We denote by R_E the set of rules generated by all ground instances of E.

Our construction is similar to the one of [3], but here the rules are oriented only by a (non-monotonic) west ordering. Hence the termination of R_E has to be insured otherwise, and therefore we require not only the left hand sides to be irreducible, but also the right hand sides.

Property 8 *Let E be a set of equations. Then for all rules $l \rightarrow r$ in R_E we have*

1. *r is irreducible by R_E*
2. *l is irreducible by $R_E \setminus \{l \rightarrow r\}$*

Proof: For the first property, by construction, if an instance e generates $l \rightarrow r$, the term r is irreducible by R_e. Since $l \succ r$, and $\succ$ is irreflexive and fulfils the subterm property, clearly $l \rightarrow r$ itself does not reduce r either. Finally, for every rule $l' \rightarrow r'$ generated by an instance d with $d \succ_{mul} e$, we must have $l' \succeq l$ and hence $l' \succ r$ which implies that l' cannot be a subterm of r either.

For the second property, by construction, if an instance e generates $l \rightarrow r$, the term l is irreducible by R_e. Now, as before, for every rule $l' \rightarrow r'$ generated by an instance d with $d \succ_{mul} e$ we must have $l' \succeq l$. Since l' must be irreducible by R_d which contains $l \rightarrow r$ we have $l' \neq l$, and hence $l' \succ l$, which implies that l' cannot be a subterm of l. □

Lemma 9 *For every set of equations E, the ground TRS R_E is convergent.*

Proof: Termination follows by Property 8.1 and Lemma 3.

For confluence, since R_E is terminating, we need only to show that R_E is locally confluent, which holds by Property 8.2. □

Theorem 10 *Let E be a set of equations closed under ordered paramodulation with respect to a west ordering $\succ$. Then $R_E^* \models E$.*

Proof: Since R_E is terminating, by Lemma 5 $\rightarrow^+_{R_E}$ is a reduction ordering. We now proceed by induction on the well-founded ordering on ground equations $(\rightarrow^+_{R_E})_{mul}$, denoted in the following by $\succ_R$. A contradiction is derived from the existence of a minimal w.r.t. $\succ_R$ ground instance e of the form $s\sigma \simeq t\sigma$ of an equation $s \simeq t$ in E such that $R^*_E \not\models e$. Since $R^*_E \not\models e$, the equation e is not a tautology of the form $u \simeq u$. Then the equation e has not generated any rule because either $s\sigma$ or $t\sigma$ is reducible by R_e. Then there exists some equation $l \simeq r$ that has generated a rule $l\sigma \rightarrow r\sigma$, such that it reduces $s\sigma$ or $t\sigma$. We consider the case where $s\sigma$ is reducible the other one is analogous. Now we have $s\sigma|_p = l\sigma$, and there are two possibilities:
(i) An inference: $s|_p$ is a non-variable position of s.
Then there exists an inference by ordered paramodulation:

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\theta}$$

whose conclusion has an instance d of the form $(s[r]_p \simeq t)\sigma$, such that $e \succ_R d$ and $R^*_E \not\models d$, contradicting the minimality of e. Note that this inference satisfies the ordering constraints of ordered paramodulation, since $l\sigma \succ r\sigma$.
(ii) Lifting: $s|_{p'}$ is a variable x for some prefix p' of p.
Then $p = p' \cdot p''$ and $x\sigma|_{p''}$ is $l\sigma$. Now let σ' be the ground substitution with the same domain as σ but where $x\sigma' = x\sigma[r\sigma]_{p''}$ and $y\sigma' = y\sigma$ for all other variables y. Then $R^*_E \not\models s\sigma' = t\sigma'$ and $e\sigma \succ_R e\sigma'$, contradicting the minimality of e. □

4.1 A slightly stronger paramodulation rule

In the proof of termination of R_E (lemma 9) we can as well use lemma 4 instead of lemma 3. More precisely, we only need the right hand sides of the rules in R_E to be irreducible at non-topmost positions (instead of being completely irreducible). Due to this observation, we can restrict our paramodulation rule avoiding its application at topmost positions of small sides of equations. Then we obtain the following *restricted ordered paramodulation* rule:

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$, the most general unifier of $s|_p$ and l, where $s|_p$ is not a variable, and l is maximal in its premise, and if $p = \lambda$ then s is also maximal in its premise. That is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$, and, if $p = \lambda$ then we also have $s\sigma\theta \succ t\sigma\theta$.

Now to prove theorem 10, we slightly modify Definition 7, the generation of R_E, obtaining the following variant of it:

Definition 11 *Let E be a set of equations. An instance e of the form $l \simeq r$ of an equation in E* generates *the rule $l \rightarrow r$ if*

1. *$l \succ r$,*
2. *l is irreducible by R_e, and*
3. *r is irreducible by R_e at non-topmost positions.*

where R_e is the set of rules generated by all instances d of equations in E such that $e \succ_{mul} d$. We denote by R_E the set of rules generated by all ground instances of E.

We can now adapt Property 8, and prove it analogously. From this property, lemma 9 holds as before but using lemma 4 to conclude termination.

Property 12 *Let E be a set of equations. Then for all rules $l \rightarrow r$ in R_E we have*

1. *r is irreducible by R_E at non-topmost positions.*
2. *l is irreducible by $R_E \setminus \{l \rightarrow r\}$.*

Now we adapt the proof of Theorem 10 for the restricted ordered paramodulation rule.

Theorem 13 *Let E be a set of equations closed under (restricted) ordered paramodulation with respect to a west ordering $\succ$. Then $R^*_E \models E$.*

Proof: As in Theorem 10, we proceed by induction on the well-founded ordering on ground equations $\succ_R$, deriving a contradicton from the existence of a minimal w.r.t. $\succ_R$ ground instance e of the form $s\sigma \simeq t\sigma$ of an equation $s \simeq t$ in E such that $R^*_E \not\models e$. Since $s\sigma$ cannot be equal to $t\sigma$, we assume w.l.o.g. that $s\sigma \succ t\sigma$.

The equation e has not generated any rule because $s\sigma$ is reducible by R_e or $t\sigma$ is reducible by R_e at a non-topmost position. Then there exists some equation $l \simeq r$ that has generated a rule $l\sigma \rightarrow r\sigma$, which reduces $s\sigma$ or $t\sigma$. The case where $s\sigma$ is reducible is analogous as in Theorem 10. For the other case we have $t\sigma|_p = l\sigma$, for some position $p \neq \lambda$. If p is below a variable position in t then we apply the same lifting argument as in Theorem 10, and otherwise, since $p \neq \lambda$, we can conclude as well by the existence of an ordered paramodulation inference with a smaller conclusion. □

This latter restricted ordered paramodulation rule will be adapted for the Horn case and the case of general clauses in Sections 5 and 6 respectively, and the former one will be used to obtain the results on unfailing completion in Section 7.

5 The Horn case

In this section we generalise the results of the previous section to Horn clauses. In the following inference system it is assumed that in each clause with a non-empty antecedent one of these negative equations, the one that is written underlined, has been *selected* (see, e.g., [3]). In the Horn case this leads to positive unit strategies (and in the non-Horn case to positive strategies): left premises of paramodulations are unit clauses, and the only inferences involving non-unit clauses are equality resolution or paramodulation left on its selected equation. The inference system $\mathcal{H}$ for Horn clauses with respect to the west ordering $\succ$ is defined as follows:

paramodulation right:

$$\frac{\rightarrow l \simeq r \qquad \rightarrow s \simeq t}{\rightarrow (s[r]_p \simeq t)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\rightarrow l \simeq r \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma} \qquad \text{where } \sigma = mgu(s, t)$$

where moreover in both paramodulation rules $s|_p$ is not a variable, and l is maximal in its premise, and for paramodulation right if $p = \lambda$ then s is also maximal in its premise. That is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$, and, if $p = \lambda$ in paramodulation right then also $s\sigma\theta \succ t\sigma\theta$.

Definition 14 *Let S be a set of Horn clauses. We denote by E_S the subset of all positive unit clauses in S, that is, $E_S = \{\ s \simeq t \ \mid \rightarrow s \simeq t \in S\ \}$, and by R_S the set of rules generated by E_S as in Definition 11 (i.e. $R_S = R_{E_S}$).*

We now use multiset extensions for lifting orderings $\succ$ on terms to orderings on equations and clauses. Let C be a ground clause, and let $emul(s \simeq t)$ be $\{s, t\}$ if $s \simeq t$ is a positive equation in C, and $\{s, s, t, t\}$ if it is negative. Then, if $\succ$ is an ordering, we define the ordering $\succ_e$ on (occurrences of) ground equations in a clause by $e \succ_e e'$ if $emul(e) \succ_{mul} emul(e')$. Similarly, $\succ_c$ on ground clauses is defined $C \succ_c D$ if $mse(C) \ (\succ_{mul})_{mul} \ mse(D)$, where $mse(C)$ is the multiset of all $emul(e)$ for ocurrences e of equations in C.

Theorem 15 *(refutation completeness of $\mathcal{H}$ for Horn clauses) Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\Box \in S$ if, and only if, S is unsatisfiable.*

Proof: The left to right implication is trivial. For the other one, let S be a set of clauses closed under $\mathcal{H}$ such that $\Box \notin S$. As in Theorem 10, we proceed by induction on $\succ_c^{R_S}$, which will be denoted in the following by $\succ_R$. This ordering is monotonic where needed (see the cases below) and well-founded. Again it is proved that R_S^* is a model of S by deriving a contradiction from the existence of a minimal w.r.t. $\succ_R$ ground instance C of a clause in S such that $R_S^* \not\models C$. Since E_S is closed under restricted ordered paramodulation, by Theorem 13, we have $R_S^* \models E_S$, and therefore C cannot be a positive unit clause. Two cases should be considered:

1. C is an instance $\Gamma\sigma, \underline{s\sigma = t\sigma} \rightarrow \Delta\sigma$ of a clause $\Gamma, \underline{s = t} \rightarrow \Delta$, where $s\sigma \equiv t\sigma$. Then there is an inference by equality resolution

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\theta}$$

whose conclusion has an instance D of the form $\Gamma\sigma \rightarrow \Delta\sigma$ such that $C \succ_R D$ and moreover, D is in S and $R_S^* \not\models D$, which is a contradiction.

2. C is an instance $\Gamma\sigma, \underline{s\sigma = t\sigma} \rightarrow \Delta\sigma$ of a clause $\Gamma, \underline{s = t} \rightarrow \Delta$, where $s\sigma \not\equiv t\sigma$. Then, since $R_S^* \not\models C$, we have $R_S^* \models s\sigma \simeq t\sigma$, and, since R_S is convergent, this implies that one of both terms is reducible by R_S. Then either the lifting argument applies like in Theorem 10, or else there is an inference by paramodulation left:

$$\frac{\rightarrow l \simeq t \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\theta}$$

whose conclusion has an instance D of the form $(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\sigma$ such that $C \succ_R D$ and $R_S^* \not\models D$, which is a contradiction. $\Box$

6 General Clauses

Now we consider general clauses. As for the Horn case, we consider that in each clause with a non-empty antecedent one of its negative equations, the one that is written underlined, has been selected. The inference system $\mathcal{I}$ with respect to the given west ordering $\succ$ is defined as follows:

paramodulation right:

$$\frac{\rightarrow l \simeq r, \Delta \qquad \rightarrow s \simeq t, \Delta'}{(\rightarrow s[r]_p \simeq t, \Delta, \Delta')\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\rightarrow l \simeq r, \Delta \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta'}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta, \Delta')\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma} \qquad \text{where } \sigma = mgu(s, t)$$

equality factoring:

$$\frac{\rightarrow s \simeq t, s' \simeq t', \Delta}{(t \simeq t' \rightarrow s \simeq t, \Delta)\sigma} \qquad \text{where } \sigma = mgu(s, s')$$

where in both paramodulation rules $s|_p$ is not a variable. The ordering restrictions are as follows:

In paramodulation right, l is strictly maximal in $l \simeq r$, and $l \simeq r$ is the strictly maximal equation in its premise, and if $p = \lambda$, then s is strictly maximal in $s \simeq t$ and $s \simeq t$ is the strictly maximal equation in its premise. Formally: for some ground substitution θ it holds that $l\sigma\theta \succ r\sigma\theta$ and $(l \simeq r)\sigma\theta \succ_e e\sigma\theta$ for all equations e in Δ, and if $p = \lambda$, then $s\sigma\theta \succ t\sigma\theta$ and $(s \simeq t)\sigma\theta \succ_e e\sigma\theta$ for all equations e in Δ'.

In paramodulation left, l is strictly maximal in $l \simeq r$, and $l \simeq r$ is strictly maximal in its premise, and if $p = \lambda$, then s is strictly maximal in $s \simeq t$.

Finally, in equality factoring, s is strictly maximal in $s \simeq t$ and $s \simeq t$ is maximal.

Definition 16 *Let S be a set of clauses. An instance C of the form $\to l \simeq r, \Delta$ of a clause in S* generates *the rule $l \to r$ if*

1. *$R_C^* \not\models C$,*
2. *$l \succ r$, and $(l \simeq r) \succ_e e$ for all equations e in Δ,*
3. *l is irreducible by R_C,*
4. *r and Δ are irreducible at non-topmost positions by R_C*
5. *$R_C^* \models r \simeq t$ for no equation $l \simeq t$ in Δ.*

where R_C is the set of rules generated by all instances D of clauses in S such that $C \succ_c D$. We denote by R_S the set of rules generated by all ground instances of S.

Again we have the following property which implies convergence of R_S.

Property 17 *Let S be a set of clauses. Then for all rules $l \to r$ in R_S we have*

1. *r is irreducible by R_S at non-topmost positions.*
2. *l is irreducible by $R_S \setminus \{l \to r\}$.*

Lemma 18 *Let S be a set of clauses. Then the ground TRS R_S is convergent.*

Lemma 19 *Let S be a set of clauses. If $\to l \simeq r, \Delta$ is an instance C of a clause in S that generates the rule $l \to r$ in R_S, then $R_S^* \not\models \Delta$.*

Proof: We first prove that if $R_S^* \models s \simeq t$ for ground s and t that are irreducible at non-topmost positions and such that $s \succ t$, then no rules with left hand sides greater than s (w.r.t. $\succ$) are used in the rewrite proof. We proceed by induction on the size of $s \simeq t$ w.r.t. $\succ_{mul}$. The first step can only apply at topmost position of s or t. If it is on t then we obtain $s \simeq r$, where $t \to r$ is the applied rule in R_S, and hence $s \succ t \succ r$, by definition of R_S, and r is irreducible at non-topmost positions by Property 17. 1, and then we can conclude by induction hypothises. Otherwise, we obtain $r \simeq t$, where $s \to r$ is the applied rule in R_S, and hence as before $s \succ r$ and r is irreducible at non-topmost positions. If $r = t$ we are done, and otherwise if $r \succ t$ or $t \succ r$ we can conclude by induction.

Now, assume $s \simeq t$ is a ground equation in Δ with a rewrite proof using R_S and such that $s \succ t$. The rules generated by clauses D with $D \succ_c C$ cannot be used, since they have left hand sides greater than l and hence greater than s w.r.t. $\succ$, and s and t are irreducible at non-topmost positions. But since $R_C^* \not\models s \simeq t$ the rule $l \to r$ is used. Hence $s \equiv l$ and $s \simeq t$ rewrites into $r \simeq t$, to which $l \to r$ cannot be applied any more, which contradicts the last condition of Definition 16. $\Box$

We now introduce a well-founded ordering $\succ_R$ that will be used in the proof of Theorem 23. In contrast to the Horn and equational cases, it does not coincide with $\to_{R_S}^+$.

Definition 20 *Let S be a set of clauses. By $\succ_R$ we denote the smallest transitive relation such that $s \succ_R t$ whenever (i) $s \to_{R_S}^+ t$ or (ii) $s \rhd t$ or (iii) s and t are irreducible at non-topmost positions w.r.t. R_S and $s \succ t$.*

The ordering $\succ_R$ fulfils the following properties wrt. R_S.

Property 21 *Let S be a set of clauses. Then for all ground terms s and t s.t. s is irreducible at non-topmost positions and $s \succ_R t$ we have*

1. *$s \succ t$*
2. *t is irreducible at non-topmost positions*

Proof: If $s \succ_R t$ by case (ii) or (iii) it trivially holds. Otherwise, $s \to_{R_S} s_1 \to_{R_S} \dots \to_{R_S} s_n \to_{R_S} t$. Since s is irreducible at non-topmost positions and, by property 17.1, all right hand sides of rules in R_S are irreducible at non-topmost positions, all steps in the sequence are at topmost positions, which implies on one hand that t is irreducible at non-topmost positions, and on the other, by definition of R_S, that $s \succ s_1 \succ \dots \succ s_n \succ t$. $\Box$

Lemma 22 *Let S be a set of clauses. Then $\to_{R_S}^+$ is a reduction ordering and $\succ_R$ is well-founded.*

Proof: Since R_S is a terminating TRS, $\to_{R_S}^+$ is a reduction ordering by Lemma 5. Now assume $\succ_R$ is not well-founded. Since $(\to_{R_S}^+ \cup \rhd)^+$ is well-founded by Lemma 2, there is also some such infinite sequence $t_1 \succ_R t_2 \succ_R t_3 \succ_R \dots$ starting with case (iii): where t_1 and t_2 are irreducible w.r.t. R_S at non-topmost positions and $t_1 \succ t_2$. Then, by Property 21, t_3, t_4 etc, are all irreducible at non-topmost positions, and $t_1 \succ t_2 \succ t_3 \dots$ w.r.t. the well-founded west ordering $\succ$, which is a contradiction. $\Box$

Theorem 23 *Let S be a set of clauses closed under $\mathcal{I}$ with respect to a west ordering $\succ$. Then $\square \in S$ if, and only if, S is unsatisfiable.*

Proof: We prove that if $\square \notin S$ then R_S^* is a model of S by induction on $(\succ_R)_c$, which is a well-founded ordering on clauses. In the following, we (ambiguously) write $\succ_R$ for terms, equations and clauses instead of $\succ_R$, $(\succ_R)_e$ and $(\succ_R)_c$ respectively. We derive a contradiction from the existence of a minimal w.r.t. $\succ_R$ ground instance C of a clause in S such that $R_S^* \not\models C$.

1. We first consider the case where C is an instance with σ of a positive clause $\rightarrow s \simeq t, \Delta$, where $s\sigma \simeq t\sigma$ is strictly maximal with respect to $\succ_e$ in C and w.l.o.g. $s\sigma \succ t\sigma$. Since $R_S^* \not\models C$, we know C has not generated any rule due to one of the following reasons:

1a. $t\sigma$ is reducible by R_C at a non-topmost position. Then there exists some clause $\rightarrow l \simeq r, \Delta'$ in S whose instance C' with $C \succ_c C'$ generates the rule $l\sigma \rightarrow r\sigma$ reducing $t\sigma$.

If, for some prefix p' of p, the term $t|_{p'}$ is a variable x, then the same lifting argument as in proof of Theorem 10 applies.

Otherwise, $t|_p$ is a non-variable subterm of t, and there exists an inference by paramodulation right:

$$\frac{\rightarrow l \simeq r, \Delta' \qquad \rightarrow s \simeq t, \Delta}{(\rightarrow s \simeq t[r]_p, \Delta, \Delta')\theta}$$

whose conclusion has an instance D of the form $(\rightarrow s \simeq t[r]_p, \Delta, \Delta')\sigma$ such that by Lemma 19 $R_S^* \not\models D$. This contradicts the minimality of C since $C \succ_R D$ for the following reasons: $mse(C) = mse(\Delta) \cup \{\ \{s\sigma, t\sigma\}\ \}$ and $mse(D) = mse(\Delta) \cup \{\ \{s\sigma, t\sigma[r\sigma]_p\}\ \} \cup mse(\Delta')$. Then we need to prove that $\{s\sigma, t\sigma\} \succ_{Rmul} \{s\sigma, t\sigma[r\sigma]_p\}$ and $(s\sigma \simeq t\sigma) \succ_R e\sigma$ for all equations e in Δ'. The first one holds since, by monotonicity of $\rightarrow_{R_S}^+$, we have $t\sigma[l\sigma]_p \succ_R t\sigma[r\sigma]_p$. For the second one, we have $l\sigma \succeq u\sigma$ for all terms u in Δ', and since they are also irreducible at non-topmost positions, we have $l\sigma \succeq_R u\sigma$. Furthermore $t\sigma \succ_R l\sigma$, since $t\sigma \rhd l\sigma$, which implies $t\sigma \succ_R u\sigma$ for all terms u in Δ', and hence we can conclude.

1b. $s\sigma$ is reducible by R_C and case 1a. does not apply. Then we argue as in the previous case by lifting or an inference in s whose conclusion has an instance D of the form $(\rightarrow s[r]_p \simeq t, \Delta, \Delta')\sigma$, contradicting the minimality of C. Here $C \succ_R D$ if $p \neq \lambda$ as in the previous case. If $p = \lambda$ then $(s\sigma \simeq t\sigma) \succ_R e\sigma$ for all equations e in Δ': since $C \succ_c C'$, and $s\sigma \simeq t\sigma$ is the strictly maximal equation of C, we have $(s\sigma \simeq t\sigma) \succeq_e (l\sigma \simeq r\sigma) \succ_e e\sigma$, and since all these equations are irreducible at non-topmost positions, $(s\sigma \simeq t\sigma) \succ_R e\sigma$.

1c. An equation $u\sigma \simeq v\sigma$ in Δ is reducible by R_C at a non-topmost position. The proof is like case 1a.

1d. None of the previous cases applies and Δ is of the form $s' \simeq t', \Delta'$ where $s\sigma \equiv s'\sigma$ and $R_S^* \models t\sigma \simeq t'\sigma$, that is, the last condition of Definition 16 fails. Then there exists some inference by equality factoring

$$\frac{\rightarrow s \simeq t, s' \simeq t', \Delta'}{(t \simeq t' \rightarrow s \simeq t, \Delta')\theta}$$

whose conclusion has an instance D such that $R_S^* \not\models D$. This contradicts the minimality of C: since $s\sigma$, $t\sigma$, and $t'\sigma$ are irreducible at non-topmost positions, and $s'\sigma \succ t\sigma$ and $s'\sigma \succ t'\sigma$, we have $C \succ_R D$.

2. If C is an instance with σ of $\rightarrow s \simeq t, \Delta$, where $s\sigma \simeq t\sigma$ is maximal but not strictly maximal with respect to $\succ_e$ in C, then condition 2 of Definition 16 fails. If $s\sigma \simeq t\sigma$ is reducible at non-topmost positions, then the same reasoning as in case 1a. applies. Otherwise, the proof of case 1d. applies.

3. If C is an instance with σ of a clause $\Gamma, \underline{s \simeq t} \rightarrow \Delta$, where $s\sigma \equiv t\sigma$, we conclude, as in the proof of Theorem 15, by equality resolution.

4. If C is an instance with σ of a clause $\Gamma, \underline{s \simeq t} \rightarrow \Delta$, where $s\sigma \not\equiv t\sigma$. Then $R_S^* \models s\sigma \simeq t\sigma$. Then, since R_S is convergent, there is a rewrite proof for $s\sigma \simeq t\sigma$. W.l.o.g. assume $s\sigma \succ t\sigma$. This implies that either $s\sigma$ is reducible or $t\sigma$ is reducible at a non-topmost position (otherwise the only possible reduction step is at the top of $t\sigma$ and a new term t' is obtained with $t\sigma \succ t'$ and t' again irreducible except at the top; since $s\sigma \succ t\sigma$, a sequence of topmost steps on $t\sigma$ can never produce $s\sigma$).

Then, as before, either the lifting argument applies or there is an inference by paramodulation left (we only develope the case where the step takes place in s):

$$\frac{\rightarrow l \simeq r, \Delta' \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta', \Delta)\theta}$$

whose conclusion has an instance D of the form $(\Gamma, s[r]_p \simeq t \rightarrow \Delta, \Delta')\sigma$ such that $R_S^* \not\models D$. This is a contradiction as before, since also here $C \succ_R D$ for the following reasons: we have $s\sigma \trianglerighteq l\sigma$ and hence $s\sigma \succeq_R l\sigma$. We also have $l\sigma \succ_R r\sigma$, and $l\sigma \simeq r\sigma \succ_R u\sigma \simeq v\sigma$ for all equations $u \simeq v$ in Δ' and hence $\{s\sigma, s\sigma, t\sigma, t\sigma\} \succ_{Rmul} \{l\sigma, r\sigma\} \succ_{Rmul} \{u\sigma, v\sigma\}$. $\square$

7 Unfailing completion

In this section, E denotes a set of equations and $\succ_r$ a reduction ordering on $T(\mathcal{F}, \mathcal{X})$. Then a *convergent TRS for E and $\succ_r$* is a convergent TRS, logically equivalent to E, and such that $l \succ_r r$ for all its rules $l \rightarrow r$. The problem we deal with is finding a convergent TRS for the given E and $\succ_r$ whenever it exists, and find it in finite time if it is finite.

It is not difficult to devise a procedure of theoretical nature for doing this by enumerating all equational consequences of E. If a finite convergent TRS exists, it will be found by periodically checking during the enumeration process whether the subset of orientable (with $\succ_r$) rules is convergent and entails E. Confluence can be decided by checking joinability of all critical pairs. After this, entailment of E can be decided by rewriting. Note that confluence is not affected by the unnecessary rules. Regarding practically useful procedures, Devie showed that for left- and right *linear* E (i.e. no variable occurs more than once in a side of an equation) standard Knuth-Bendix completion finds R [10]. For the general problem, the existing procedures still rely on the enumeration of all equational consequences (see e.g., [9]).

For every convergent TRS its unique *canonical*[3] version can be obtained by *interreducing* it by (i) normalizing all right hand sides and (ii) removing rules whose left hand sides are reducible by other rules [8]. In the following, R will denote the canonical TRS for E and $\succ_r$ (we assume R exists), and we denote by R_g the canonical TRS for $gnd(R)$, the set of all ground instances of rules of R.

The following lemma strengthens the uniqueness result of [8] to TRS included in a west ordering, but only for the ground case:

Lemma 24 *Let $\succ$ be a west ordering, and let R_1 and R_2 be interreduced TRS over $T(\mathcal{F})$, both included in $\succ$, and such that $R_1^* = R_2^*$. Then $R_1 = R_2$.*

Proof: Consider the rule $l \to r$ in $(R_1 \cup R_2) \setminus (R_1 \cap R_2)$ with minimal l w.r.t. $\succ$. Assume w.l.o.g. $l \to r \in R_1$. Since $R_1^* \models l \simeq r$ and hence $R_2^* \models l \simeq r$ and R_2 is convergent, there must be a rewrite proof by R_2 for $l \simeq r$. But, since R_1 is interreduced, r and all strict subterms of l are irreducible w.r.t. R_1, and hence also w.r.t. R_2, since any rule reducing them has a lhs smaller than l w.r.t. $\succ$. Hence there is a rule $l \to r'$ in R_2. But then r' and r are both irreducible by R_2 and $R_2 \models r \simeq r'$. This implies that r and r' are the same term, contradicting the assumption that $l \to r \notin R_2$. $\Box$

In the following we consider the ordered equational paramodulation rule given in Definition 6 w.r.t. a west ordering $\succ$ extending $\succ_r$, and the ground TRS R_E as in Definition 7. Then, by Property 8, for all rules $l \to r$ in R_E we have that $l \succ r$ and that its sides l and r are both irreducible by R_E.

Lemma 25 *Let E' be the closure of E under ordered equational paramodulation w.r.t. a west ordering $\succ$ extending $\succ_r$. Then $R_g = R_{E'}$.*

[3]Unfortunately, the word canonical is sometimes also used as equivalent of convergent.

Proof: By Theorem 10 we have $R_{E'}^* \models E'$, and hence $R_{E'}^* \models E$, and hence $E^* = R_{E'}^*$ since all rules in $R_{E'}$ are logical consequences of E. Furthermore, we clearly have $E^* = R^* = R_g^*$, and hence $R_g^* = R_{E'}^*$. We also have $R_g \subseteq \succ_r \subseteq \succ$, and $R_{E'} \subseteq \succ$. Therefore, since both are interreduced we conclude by Lemma 24 that $R_{E'} = R_g$. $\Box$

Theorem 26 *Let E' be the closure of E under equational ordered paramodulation w.r.t. a west ordering $\succ$ extending $\succ_r$. Then $E' \supseteq R$.*

Proof: W.l.o.g. assume there are sufficiently new constants in $\mathcal{F}$ that do not occur in E or R. Let $l \to r$ be an arbitrary rule in R. We prove that $l \to r \in E'$. Let σ be the ground substitution replacing each variable with a distinct new constant. Then $l\sigma \to r\sigma$ is in $gnd(R)$ and also in R_g: since R is interreduced, r is irreducible w.r.t. R (and hence $r\sigma$ by $gnd(R)$), and l is irreducible w.r.t. $R \setminus \{l \to r\}$ (and hence $l\sigma$ by $gnd(R) \setminus \{l\sigma \to r\sigma\}$). If $l\sigma \to r\sigma$ is in R_g and by the previous lemma $R_g = R_{E'}$, then $l\sigma \to r\sigma$ is in $R_{E'}$. Since the new constants do not occur in E, if $l\sigma \to r\sigma \in R_{E'}$, then some $l' \simeq r'$ is in E', and $l\sigma \equiv l'\theta\sigma$ and $r\sigma \equiv r'\theta\sigma$ for some θ. We conclude by showing that θ is the identity substitution: there is no equation $l' \simeq r'$ that strictly subsumes a rule in R; otherwise, the rewrite proof by R of $l' \simeq r'$ would apply some rule different from $l \to r$, which would then also reduce $l \to r$, contradicting $l \to r \in R$. $\Box$

8 Future work

Simplification and deletion techniques that are compatible with refutation completeness can be uniformly covered in the model generation technique by notions of *redundancy* for inferences and clauses [3], where *saturation* amounts to the closure under $\mathcal{I}$ *up to redundancy*. In our setting, we work with two different orderings. The inferences are computed w.r.t. the west ordering $\succ$, and the completeness proof uses the reduction ordering $\succ_R$. Unfortunately, redundancy should hence be defined w.r.t. $\succ_R$, which is unknown during the saturation process. Therefore it has to be more carefully explored to what extent the redundancy notions are applicable. In many cases it is clear that $\succ_R$ can be sufficiently approximated. For example, practical redundancy notions like tautology deletion or subsumption are obviously correct.

Regarding our completion procedure, we are currently exploring whether simplification by rewriting with respect to the reduction ordering $\succ_r$ is correct, and whether inferences with (or on) small sides w.r.t. $\succ_r$ are needed.

We are also working on more restrictive inference systems. Inferences on non-maximal positive atoms are needed: if the inconsistent set S consists of the clauses

$$\begin{array}{rl} & \rightarrow P(b,b), P(a,b) \\ \underline{P(b,b)}, P(a,b) & \rightarrow \\ & \rightarrow a \simeq b \end{array}$$

with a west ordering $\succ$ where $a \succ b$ and $P(b,b) \succ P(a,b)$, then the only other possible inference produces the tautology $P(a,b) \rightarrow P(a,b)$.

Also, inferences on small sides of positive equations are needed: if the inconsistent S consists of the clauses

$$\begin{array}{rl} & \rightarrow a \simeq b \\ & \rightarrow g(b) \simeq g(g(a)) \\ x \simeq g(x) & \rightarrow \end{array}$$

with a west ordering $\succ$ where $a \succ b$ and $g(b) \succ g(g(a))$, then the only other possible inferences produce clauses of the form $b \simeq g^n(b) \rightarrow$ and $b \simeq g^n(a) \rightarrow$ for $n > 1$.

Ordered paramodulation is a very adequate inference rule for dealing with constrained clauses and the *basic* strategy. It seems that these ideas are directly applicable here: subterms created by unifiers of inferences on ancestors can be *blocked* for inferences, and in our setting this is true also for proper subterms of the term $r\sigma$ (which are irreducible by R_S in our proof) in the conclusions of paramodulation inferences, see [4, 14] for details.

Finally, we are of course also exploring completeness results for deduction modulo equational theories E for which monotonic E-compatible total orderings do not exist, but that can now be handled with our techniques.

References

[1] L. Bachmair and N. Dershowitz. Equational inference, canonical proofs, and proof orderings. *J. of the Association for Computing Machinery*, 41(2):236–276, Feb. 1994.

[2] L. Bachmair, N. Dershowitz, and J. Hsiang. Orderings for equational proofs. In *[First Annual] Symposium on Logic in Computer Science*, pages 346–357, Cambridge, Massachusetts, USA, June 16–18, 1986. IEEE Computer Society Press.

[3] L. Bachmair and H. Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation*, 4(3):217–247, 1994.

[4] L. Bachmair, H. Ganzinger, C. Lynch, and W. Snyder. Basic paramodulation. *Information and Computation*, 121(2):172–192, 1995.

[5] D. Basin and H. Ganzinger. Complexity Analysis Based on Ordered Resolution. In *Eleventh Annual IEEE Symposium on Logic in Computer Science*, pages 456–465, New Brunswick, New Jersey, USA, July 27–30, 1996. IEEE Computer Society Press.

[6] N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, chapter 6, pages 244–320. Elsevier Science Publishers B.V., Amsterdam, New York, Oxford, Tokyo, 1990.

[7] N. Dershowitz and Z. Manna. Proving termnation with multiset orderings. *Comm. of ACM*, 22(8), 1979.

[8] N. Dershowitz, L. Marcus, and A. Tarlecki. Existence, uniqueness, and construction of rewrite systems. *SIAM Journal on Computing*, 17(4):629–639, 1988.

[9] H. Devie. PhD thesis, Université de Paris-Sud, Orsay, France.

[10] H. Devie. Linear completion. In S. Kaplan and M. Okada, editors, *Conditional and Typed Rewriting Systems, 2nd International Workshop*, LNCS 516, pages 233–245, Montreal, Canada, June 11–14, 1990. Springer-Verlag.

[11] J. Hsiang and M. Rusinowitch. Proving refutational completeness of theorem proving strategies: the transfinite semantic tree method. *Journal of the ACM*, 38(3):559–587, jul 1991.

[12] C. Kirchner, H. Kirchner, and M. Rusinowitch. Deduction with symbolic constraints. *Revue Française d'Intelligence Artificielle*, 4(3):9–52, 1990.

[13] R. Nieuwenhuis. Decidability and complexity analysis by basic paramodulation. *Information and Computation*, 147:1–21, 1998. Extended abstract in IEEE LICS'96.

[14] R. Nieuwenhuis and A. Rubio. Theorem Proving with Ordering and Equality Constrained Clauses. *Journal of Symbolic Computation*, 19(4):321–351, April 1995.

[15] W. Wechler. *Universal Algebra for Computer Scientists*, volume 25 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1991.

[16] C. Weidenbach. SPASS—version 0.49. *Journal of Automated Reasoning*, 18(2):247–252, Apr. 1997.