Mining Implications from Lattices of Closed Trees

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Introduction

Problem

Given a dataset $\mathcal{D}$ of rooted, unlabelled and unordered trees, find a “basis”: a set of rules that are sufficient to infer all the rules that hold in the dataset $\mathcal{D}$. 
Set of Rules:

\[ A \rightarrow \Gamma_{\emptyset}(A). \]

- Antecedents are obtained through a computation akin to a hypergraph transversal.
- Consequents follow from an application of the closure operators.
Set of Rules:

\[ A \rightarrow \Gamma_{\square}(A) . \]

Diagram:
Trees

Our trees are:
- Rooted
- Unlabeled
- Unordered

Our subtrees are:
- Induced
- Top-down

Two different ordered trees but the same unordered tree
Deterministic association rules

- Logical implications are the traditional mean of representing knowledge in formal AI systems. In the field of data mining they are known as **association rules**.

<table>
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<tr>
<th>M</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
<tr>
<td>$m_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
<td>$m_2$</td>
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<td>$m_3$</td>
<td>0</td>
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$a \rightarrow b, d$

d \rightarrow b

a, b \rightarrow d

- Deterministic association rules are implications with 100% confidence.

- An advantage of deterministic association rules is that they can be studied in purely logical terms with **propositional Horn logic**.
Propositional Horn Logic

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- Assume a finite number of variables.
  - $V = \{a, b, c, d\}$
- A clause is **Horn** iff it contains at most one positive literal.
  - $\overline{a} \lor \overline{b} \lor d$  $a, b \rightarrow d$
- A **model** is a complete truth assignment from variables to $\{0, 1\}$.
  - $m(a) = 0, m(b) = 1, m(c) = 1, \ldots$
- Given a set of models $M$, the Horn theory of $M$ corresponds to the conjunction of all Horn clauses satisfied by all models from $M$. 

$a \rightarrow b, d$  $(\overline{a} \lor b) \land (\overline{a} \lor d)$ 

$d \rightarrow b$  $\overline{d} \lor b$

$a, b \rightarrow d$  $\overline{a} \lor \overline{b} \lor d$
Theorem

Given a set of models $M$, there is exactly one minimal Horn theory containing it. Semantically, it contains all the models that are intersections of models of $M$. This is sometimes called the empirical Horn approximation.

We propose

- Closure operator
- translation of tree set of rules to a specific propositional theory
Closure Operator

- \( \mathcal{D} \): the finite input dataset of trees
- \( \mathcal{T} \): the (infinite) set of all trees

Definition

We define the following the Galois connection pair:

- For finite \( A \subseteq \mathcal{D} \)
  - \( \sigma(A) \) is the set of subtrees of the \( A \) trees in \( \mathcal{T} \)
    
    \[ \sigma(A) = \{ t \in \mathcal{T} \mid \forall t' \in A \ (t \preceq t') \} \]

- For finite \( B \subset \mathcal{T} \)
  - \( \tau_\mathcal{D}(B) \) is the set of supertrees of the \( B \) trees in \( \mathcal{D} \)
    
    \[ \tau_\mathcal{D}(B) = \{ t' \in \mathcal{D} \mid \forall t \in B \ (t \preceq t') \} \]

Closure Operator

The composition \( \Gamma_\mathcal{D} = \sigma \circ \tau_\mathcal{D} \) is a closure operator.
Galois Lattice of closed set of trees
Intuition

- One propositional variable $v_t$ is assigned to each possible subtree $t$.
- A set of trees $A$ corresponds in a natural way to a model $m_A$.
- Let $m_A$ be a model: we impose on $m_A$ the constraints that if $m_A(v_t) = 1$ for a variable $v_t$, then $m_A(v_{t'}) = 1$ for all those variables $v_{t'}$ such that $v_{t'}$ represents a subtree of the tree represented by $v_t$.

$$\mathcal{R}_0 = \{ v_{t'} \rightarrow v_t \mid t' \preceq t, t \in \mathcal{U}, t' \in \mathcal{U} \}$$
Theorem

The following propositional formulas are logically equivalent:

- the conjunction of all the Horn formulas that are satisfied by all the models $m_t$ for $t \in \mathcal{D}$
- the conjunction of $R_0$ and all the propositional translations of the formulas in $R'_\mathcal{D}$

$$R'_\mathcal{D} = \bigcup_{\mathcal{C}} \{ A \rightarrow t \mid \Gamma_{\mathcal{D}}(A) = \mathcal{C}, t \in \mathcal{C} \}$$

- the conjunction of $R_0$ and all the propositional translations of the formulas in a subset of $R'_\mathcal{D}$ obtained transversing the hypergraph of differences between the nodes of the lattice.
Association Rule Computation Example
Association Rule Computation Example
Association Rule Computation Example
Given three trees $t_1$, $t_2$, $t_3$, we say that $t_1 \land t_2 \rightarrow t_3$, is an implicit Horn rule (abbreviately, an implicit rule) if for every tree $t$ it holds

\[ t_1 \preceq t \land t_2 \preceq t \iff t_3 \preceq t. \]

$t_1$ and $t_2$ have implicit rules if $t_1 \land t_2 \rightarrow t$ is an implicit rule for some $t$. 
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**Implicit rules**

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\[ \text{NOT Implicit Rule} \]
Implicit rules

This supertree of the antecedents is **NOT** a supertree of the consequents.

**NOT** Implicit Rule

\[ \wedge \rightarrow \]
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$t_1$ and $t_2$ have implicit rules if $t_1 \land t_2 \rightarrow t$ is an implicit rule for some $t$. 
Theorem

All trees $a, b$ such that $a \preceq b$ have implicit rules.

Theorem

Suppose that $b$ has only one component. Then they have implicit rules if and only if $a$ has a maximum component which is a subtree of the component of $b$.

- for all $i < n$

$$a_i \preceq a_n \preceq b_1$$
Experimental Validation: CSLOGS

Number of rules
Number of rules not implicit
Number of detected rules

Support

0 5000 10000 15000 20000 25000 30000
Conclusions

- A way of extracting high-confidence association rules from datasets consisting of unlabeled trees
  - antecedents are obtained through a computation akin to a hypergraph transversal
  - consequents follow from an application of the closure operators
- Detection of some cases of implicit rules: rules that always hold, independently of the dataset