Theory of Computation **Partial Exam 2**

05/06/2026

Instructions

On the desk you are only allowed to have: this exam, black/blue ink pens, and your DNI (or UPC identification). **Electronic devices and backpacks are not allowed near the desk or your person.**

Login into the computer with
user: `XXX`
password: `XXXXXXXX`

Go to the RACSO webpage (`https://racso.cs.upc.edu/`) and login with your credentials (as usual, user: FIB username, password: DNI without letter) and wait for the exam to start.
The RACSO part of the exam starts automatically at **11:45 h**, and ends at **13:45 h** (except for students with adaptations).

This exam consists of three parts:

- **4 exercises on RACSO** (1.1 point per exercise, automatically evaluated). Each *rejected submission* removes 0.1 points if eventually the solution is accepted. Otherwise, the score for the exercise is 0 points (there are no "negative points").
- **8 multiple choice questions** (0.6 points per question).
- **2 open questions** (1 point per question).

The maximum score attainable in this exam is 11.2 out of 10.

14:30 h: this exam ends (except for students with adaptations).

Make sure to write your name/surname/DNI on every page (except this one).
Feel free to detach this page from the rest and use it to write down the calculations you might need.

Theory of Computation **Partial Exam 2**

05/06/2026 multiple-choice questions

Name: ______________ **Surname(s):** ______________________________ **DNI:** ______________

Write your answers below with blue or black ink.

Question	1	2	3	4	5	6	7	8	Total
Answer									

each correct answer: $+0.6$ points
each wrong answer: -0.1 points

1 Consider the language

$$L = \{w \in \{0,1\}^* \mid \text{value}_2(w) \in 2\mathbb{N}\} \setminus \{1^n 0^n \mid n \geq 0\}.$$

What can we say about L?

A. That $\overline{L}$ is a language accepted by a pushdown automaton but L is not.

B. That neither L nor $\overline{L}$ are languages accepted by pushdown automata.

C. That L is a language accepted by a pushdown automaton but $\overline{L}$ is not.

D. That both L and $\overline{L}$ are languages accepted by pushdown automata.

Note that the set difference is defined as $A \setminus B = A \cap \overline{B}$ for any sets A and B.

Solution: Context-free languages are exactly the ones accepted by a pushdown automaton. Both $\{1^n 0^n \mid n \in \mathbb{N}\}$ and its complement are context-free languages. Hence L is the intersection of a context-free language with a regular language, and therefore context-free. $\overline{L}$ is the union of a context-free language and a regular one, hence also context-free.

2 A Turing machine is called *right-moving* if its head can never move to the left or stay at the same position (that is, it can only move to the right). What is the class of languages accepted by right-moving Turing machines?

A. The languages in $\mathbf{P}$.

B. The context-free languages.

C. The recursive languages.

D. The regular languages.

Solution: This type of right-moving TM can only scan the input from left to right and possibly write in tape cells it will never have access to. This is equivalent to a DFA, and hence the accepted languages are exactly the regular ones.

3 Let M be a Turing machine and $A = \mathscr{L}(M)$ the language recognized by M. Let $B = \{w \mid M(w)\uparrow\}$ be the set of strings for which M does not halt. Which of the following statements is false for some M?

A. If B is cofinite, then $A \in \mathbf{R}$.

B. If B is finite, then $A \in \mathbf{R}$.

C. $A \in \mathbf{RE}$ independently of B.

D. If B is infinite, then $A \in \mathbf{R}$.

Recall that $\mathbf{R}$ is the set of all decidable languages. Also, a set S is *cofinite* if and only if $\overline{S}$ is finite.

Solution: Option D is false, consider $\mathtt{K} = \{p \mid M_p(p)\downarrow\}$. $\mathtt{K} \in \mathbf{RE}$, hence there is a TM M s.t. $\mathtt{K} = \mathscr{L}(M)$. We know that $\mathtt{K} \notin \mathbf{R}$. Take $A = \mathtt{K}$. Clearly, B is infinite, because there are infinitely many TMs that do not terminate and on them M cannot terminate.

4 Consider the language $L = \{p \mid \text{Im}(\varphi_p) \text{ is finite}\}$. Which of the following is true?

A. $L \in \mathbf{coRE} \setminus \mathbf{R}$

B. $L \in \mathbf{R}$

C. $L \in \mathbf{RE} \setminus \mathbf{R}$

D. $L \notin \mathbf{RE} \cup \mathbf{coRE}$

Solution: Option D is true: it is enough to show that $\overline{\mathtt{K}} \leq_m L$ and $\mathtt{K} \leq_m L$. We give the two reductions with the RACSO syntax. For the first reduction consider:

```
input y {
    runmxx;
    output y;
}
```

For the second reduction consider:

```
input y{
    if (mxxstopsininputsteps)
        output 1;
    output y;
}
```

We omit the simple check that the reductions are indeed correct.

5 A *vertex cover* of a graph G is a subset of its vertices that includes at least one endpoint of every edge in G. Consider the EXACT VERTEX COVER problem: "given a graph G and an integer k, is the *minimum* size of a vertex cover of G exactly k?"

With the theory seen in this course you can prove that three out of the four following sentences are true. Which is the remaining one?

A. EXACT VERTEX COVER $\in \Sigma_1^p \cup \Pi_1^p$.

B. The complement of EXACT VERTEX COVER belongs to Π_2^p.

C. EXACT VERTEX COVER $\in \Sigma_3^p$.

D. If EXACT VERTEX COVER $\notin \Sigma_1^p$, then $\Sigma_1^p \neq \Pi_1^p$.

> **Solution:** EXACT VERTEX COVER $\in \Sigma_2^p$. Which implies, taking the complement, that B is true. Since $\Sigma_2^p \subseteq \Sigma_3^p$, then C is also true. If $\Sigma_1^p = \Pi_1^p$, then the whole Polynomial Hierachy collapses to Σ_1^p and hence, in particular, EXACT VERTEX COVER $\in \Sigma_1^p$. In other words, D is also true.

6 Which of the following statements is correct?

A. If $B \in \mathbf{P}$ and $A \leq_m B$ then $A \in \mathbf{P}$.

B. If $A \in \mathbf{R}$, there is a language $B \in \mathbf{P}$ such that $A \leq_m B$.

C. If $B \in \mathbf{P}$ and $A \leq_m B$, then A might not belong to $\mathbf{R}$.

D. If $B \in \mathbf{NP}$ and $A \leq_m B$ then $A \in \mathbf{NP}$.

Recall that $\mathbf{P}$ is the set of all languages decidable in polynomial time and that $A \leq_m B$ means that there is a many-one reduction (not necessarily polynomial-time) from A to B.

> **Solution:** Option B is true: take $B = \{0, 1\}$, the reduction from A to B is simply deciding A and returning the answer.
>
> Option A is false: take as A any decision problem in $\mathbf{EXP} \setminus \mathbf{P}$ and $B = \{0, 1\}$. A reduction $A \leq_m B$ could be just resolving A taking exponential time and returning the answer. For option D the reasoning is similar, take as A a language that can be solved with doubly exponential time but not in $\mathbf{EXP}$ (it exists by the time hierarchy theorem).
>
> Option C is also false: if $A \leq_m B$ and $A \notin \mathbf{R}$ then $B \notin \mathbf{R}$.

7 Most computer scientists believe that $\mathbf{P} \neq \mathbf{NP}$. Under this belief, one of the following sentences is false. Which one?

A. There is some planar graph which is not 3-colorable.

B. Given a graph G and a 4-coloring of it, deciding whether G is 3-colorable is a problem in $\mathbf{P}$.

C. Given a graph G with n vertices, deciding whether G is n-colorable is a problem in $\mathbf{P}$.

D. Given a graph G and a 4-coloring of it, deciding whether G is 3-colorable is $\mathbf{NP}$-hard.

Recall that all planar graphs are 4-colorable and the coloring can be computed in polynomial time. Moreover, given a planar graph G, deciding whether G is 3-colorable is known to be $\mathbf{NP}$-hard.

> **Solution:** Option A is true: take for instance the clique with 4 vertices.
>
> Option C is true: the decision problem is trivial, every graph with n vertices is n colorable, just color each vertex with a different color.
>
> Option D is true: reduce 3-colorability of planar graphs to the language in D. The reduction is trivial, the input graph G is mapped to (G, c) where c is a 4-coloring of G (which can be computed in poly-time).
>
> Option B is false: since D is true and we are working under the assumption that $\mathbf{P} \neq \mathbf{NP}$, then B must be false.

8 Suppose all problems in $\mathbf{NP}$ are either in $\mathbf{P}$ or $\mathbf{NP}$-complete. Which of the following statements is true?

A. $\mathbf{P} = \mathbf{NP}$ and $\mathbf{NP} = \mathbf{EXP}$

B. $\mathbf{P} \neq \mathbf{NP}$ and $\mathbf{NP} = \mathbf{EXP}$

C. $\mathbf{P} = \mathbf{NP}$ and $\mathbf{NP} \neq \mathbf{EXP}$

D. $\mathbf{P} \neq \mathbf{NP}$ and $\mathbf{NP} \neq \mathbf{EXP}$

> **Solution:** $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$ and at least one of the inclusions is strict. The assumption of the exercise, by Ladner's theorem, implies that $\mathbf{P} = \mathbf{NP}$. Hence it must be also that $\mathbf{NP} \neq \mathbf{EXP}$.

Feel free to use the rest of this page and its back to write down the calculations you might need.

Name: ____________ **Surname(s):** ______________________________ **DNI:** ____________

Please use **only** the space below (or the back of the page) for your solution of the following exercise. Use blue or black ink. You can write your solution either in Catalan, Spanish, or English.

1 (1 point) Prove that your solution in exercise 3 or exercise 4 of the RACSO part of the exam is indeed a correct many-one reduction.

Solution: The solution of this exercise depends on the actual reduction used in the RACSO part of the exam.

Name: ______________ **Surname(s):** ______________________________________ **DNI:** ______________

Please use **only** the space below (or the back of the page) for your solution of the following exercise. Use blue or black ink. You can write your solution either in Catalan, Spanish, or English.

2 (1 point) Consider the decision problem `True-SAT`: given as input a CNF formula F that is true mapping all variables to true, is there another assignment satisfying F?

Show that `True-SAT` is $\mathbf{NP}$-complete.

Solution: For a CNF formula F, a certificate for it to belong to `True-SAT` is simply an assignment α different from the all 1s assignment. The certificate has size n, the number of variables of F and clearly it is polynomial in the size of F. Verifying that $F(\alpha) = 1$ is also polynomial time (actually linear), the same verifier for `SAT` would do. Therefore, $\texttt{True-SAT} \in \mathbf{NP}$.

To show that `True-SAT` is $\mathbf{NP}$-complete we show that $\texttt{CNF-SAT} \leq_m^p \texttt{True-SAT}$. Consider the many-one reduction f mapping a CNF formula $F = C_1 \wedge \cdots \wedge C_m$ in the variables $x_1, \ldots, x_n$ to the propositional formula

$$(\neg y \to (C_1 \wedge \cdots \wedge C_m)) \wedge (y \to (x_1 \wedge \cdots \wedge x_n)) \;, \tag{1}$$

but written in CNF form as

$$f(F) = \bigwedge_{j=1}^{m} (y \vee C_j) \wedge \bigwedge_{i=1}^{n} (\neg y \vee x_i) \;.$$

The CNF form of $f(F)$ is derived from (1) by just applying the definition of the implication $\to$ and distributivity properties of $\wedge$ and $\vee$.

We have that $f(F)$ is a CNF with $n + m$ clauses in $n + 1$ variables and clearly can be computed in time polynomial in the size of F. To show the correctness we have two cases to consider:

- If $F = C_1 \wedge \cdots \wedge C_m \in \texttt{CNF-SAT}$, then there is an assignment $x_1 = b_1, \ldots, x_n = b_n$ with each $b_i \in \{0, 1\}$ satisfying all the clauses C_i in F. That is, $f(F)$ has at least two satisfying assignments: $y = 1, x_1 = 1, \ldots, x_n = 1$, and $y = 0, x_1 = b_1, \ldots, x_n = b_n$. The fact that the two assignments satisfy $f(F)$ is immediate when we see $f(F)$ as the equivalent formula in (1). In other words $f(F) \in \texttt{True-SAT}$.
- If $F \notin \texttt{CNF-SAT}$, then $F = C_1 \wedge \cdots \wedge C_m$ does not have a satisfying assignment, hence the only possible way to satisfy the first implication of (1) is to have $y = 1$. Under this choice of y, the second implication of (1) becomes true when $x_1 = x_2 = \cdots = x_n = 1$. And this is the only assignment satisfying $f(F)$, hence $f(F) \notin \texttt{True-SAT}$.