20. Discuss the irreducibility and the periodicity of the following Markov chains:

$$(a)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix};(b)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix};(c)\ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$(d)\ P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 0 \end{pmatrix};(e)\ P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix};$$

21. A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that state 3 is absorbing and, starting from state 1, find the expected time until absorption occurs.

22. A fair coin is tossed repeatedly and independently. Use a Markov chain to find the expected number of tosses until the pattern HTH appears.

23. Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix:

$$P = \begin{bmatrix} 0 & 3/10 & 1/10 & 3/5 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/10 & 7/10 & 1/10 & 1/10 \\ 9/10 & 1/10 & 0 & 0 \end{bmatrix}$$

   (a) Find the stationary distribution of the Markov Chain.
   (b) Find the probability of being in state 3 after 32 steps if the chain begins at state 0.
   (c) Find the probability of being in state 3 after 128 steps if the chain begins at a state chosen uniformly at random from the four states.
   (d) Suppose that the chain begins in state 0. What is the smallest value of t for which $\max_i |P^t_{0,i} - \pi^*_i| \leq 0.01$? , where π^*_i is the stationary distribution.

24. I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

   (a) If the probability of rain is p, what is the probability that I get wet?
   (b) If the current forecast shows a $p = 0.6$, how many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

   (*Hint: Use a MC*)

25. Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.

   (a) Why is this process a Markov chain?

(b) What is the state space?

(c) Is it irreducible? Is it aperiodic?

(d) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?

(e) Which are the most likely states in steady-state? Which are the least likely ones?

26. Consider a Markov chain with states $S = \{0, ..., N\}$ and transition probabilities $p_{i,i+1} = p$ and $p_{i,i-1} = q$, for $1 \leq i \leq N-1$, where $p + q = 1$, $0 < p < 1$; assume $p_{0,1} = 1$, and $p_{N,N-1} = 1$.

    (a) Draw the MC associated graph.

    (b) Is the Markov chain irreducible?

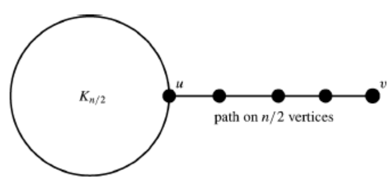
    (c) Is it aperiodic?

    (d) What is the period of the chain?

    (e) Find the stationary distribution.

27. A cat and a mouse take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $(m^2 n)$ on the expected time before the cat eats the mouse. (Hint: Consider a Markov chain whose states are the ordered pair (a, b), where a is the position of the cat and b is a position of the mouse.)

28. The lollipop graph on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices, as shown in the figure below. The node u is a part of both the clique and the path.



    Let v denote the other end of the path.

    (a) Show that the expected covering time of a random walk starting at v is $O(n^2)$.

    (b) Show that the expected covering time for a random walk starting at u is $O(n^3)$.