11. An urn contains 11 balls, 3 white, 3 red, and 5 blue balls. Take out 3 balls at random, without replacement. You win 1€, for each red ball you select and lose a 1€, for each white ball you select. Determine the probaility mass function of the amount you win. Give the value of $\mathbf{E}[X]$

12. (a) Suppose that we roll twice a fair k-sided die with the numbers 1 through k on the die's faces, obtaining values X_1 and X_2. What is $\mathbf{E}[\max(X_1, X_2)]$? What is $\mathbf{E}[\min(X_1, X_2)]$?

    (b) Show from your calculation in part (a) that $\mathbf{E}[\max(X_1, X_2)] + \mathbf{E}[\min(X_1, X_2)] = \mathbf{E}[X_1] + \mathbf{E}[X_2]$

    (c) Explain why the equation in part (b) must be true by using the linearity of expectations instead of a direct computation.

13. Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

    (a) What is the probability that $X = Y$?

    (b) What is $\mathbf{E}[\max(X, Y)]$?

    (c) What is $\mathbf{Pr}[\min(X, Y) = k]$?

    (d) What is $\mathbf{E}[X|X \leq Y]$?

14. The following approach is often called **reservoir sampling**. Suppose we have a sequence of items passing by one at a time. We want to maintain a sample of one item with the property that it is uniformly distributed over all the items that we have seen at each step. Moreover, we want to accomplish this without knowing the total number of items in advance or storing all of the items that we see. Consider the following algorithm, which stores just one item in memory at all times. When the first item appears, it is stored in the memory. When the k-th item appears, it replaces the item in memory with probability $1/k$ Explain why this algorithm solves the problem.
    The importance of the problem is following : Suppose, you are monitoring a twitter feed and you want to generate a perfectly random sample of k tweets, as that is the maximum you can store in your memory. For instance, this sample can be used for estimating the percentage of tweets on a particular subject.

15. Let $a_1, a_2, \ldots, a_n$ a sequence of n integers without repetitions. We say that a_i i a_j are *inverted* when $i < j$ but $a_i > a_j$. The BUBBLE SORT algorithm interchanges pairs of inverted elements until the sequence became sorted.

    Assume that the input to Bubble sort is is a random permutation selected uar and that X is a random variable counting the number of interchanges during the execution of the algorithm.

(a) Compute $\mathbf{E}[X]$ and $\mathbf{Var}[X]$.

(b) X is concentrated around its mean?

16. Let the random variable X be representable as a sum of random variables $X = \sum_{i=1}^{n} X_i$. Show that, if $\mathbf{E}[X_i X_j] = \mathbf{E}[X_i]\,\mathbf{E}[X_j]$ for every pair of i and j with $1 \le i < j \le n$, then $\mathbf{Var}[X] = \sum_{i=1}^{n} \mathbf{Var}[X_i]$.

17. Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive or $\oplus$ of the i-th pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ the number of Y_i that equal 1.

    (a) Show that each $Y_i = 0$ with prob $= 1/2$ (therefore, $Y_i = 1$ with probability also $1/2$)

    (b) Show that Y_i are not mutually independent.

    (c) Show that $\mathbf{E}[Y_i Y_j] = \mathbf{E}[Y_i]\,\mathbf{E}[Y_j]$.

    (d) Find $\mathbf{Var}[Y]$.

    (e) Use Chebyshev to bound $\mathbf{Pr}[|Y - \mathbf{E}[Y]| \ge n]$.

18. Let us consider a collection of points in the unit square ($[0,1]^2$). Divide the unit square into $n/\log^2 n$ square boxes. Given $\epsilon \in (0,1)$, we say that the collection of points is ϵ-nice if each ox contains at least $(1-\epsilon)\log^2 n$ points and at most $(1+\epsilon)\log^2 n$ points. Using Markov's or Chebyshev's inequalites its is possible to show that, for any $\epsilon \in (0,1)$, whp a collection of n points taken uar in the unit square is ϵ-nice?

19. Suppose that we can obtain independent samples $X_1, X_2, \ldots X_n$ of a random variable X and that we want to use these samples to estimate $\mathbf{E}[X]$. Using t samples, we use $(\sum_{i=1}^{n} X_i/t)$ for our estimate of $\mathbf{E}[X]$. We want the estimate to be within $\epsilon\mathbf{E}[X]$ from the true value of $\mathbf{E}[X]$ with probability $\ge (1-\delta)$.

    We develop an alternative approach that requires only having a bound on the $\mathbf{Var}[X]$. Let $r = \sqrt{\mathbf{Var}[X]}/\mathbf{E}[X]$.

    (a) Show using Chebyshev that $O(r^2/\epsilon^2\delta)$ samples are sufficient to solve the problem.

    (b) Suppose that we need only a weak estimate that is within $\epsilon\mathbf{E}[X]$ of $\mathbf{E}[X]$, with probability at least $3/4$. Argue that. $O(r^2/\epsilon^2)$ samples are enough for this weak estimate.

    (c) Show that, by taking the median of $O(\lg(1/\delta))$ weak estimates, we can obtain an estimate within $\epsilon\mathbf{E}[X]$ of $\mathbf{E}[X]$ with probability at least $(1-\delta)$. Conclude that we need only $(r^2\lg(1/\delta)/\epsilon^2)$ samples.