Random variables and expectation

RA-MIRI QT Curs 2020-2021

Random variables

Flip 100 times a fair coin, each time if the outcome is H we give 1€, if it is T we get 1€. At the end, how much did we win or loose?. Notice $\Omega = \{0,1\}^{100}$

Given Ω, a random variable is a function $X : \Omega \to \mathbb{R}$.
X can be interpreted as a quantity, whose value depends on the outcome of the experiment.
For ex. in the previous example, define X to be the number wins-looses $(\#1 - \#0)$.

Events $\leftrightarrow$ random variables

$\leftarrow$ Given a random variable X on Ω and $a \in \mathbb{R}$ the event $X \geq a$ represents the set $\{x \in \Omega | X(x) \geq a\}$.

$$\mathbf{Pr}[X \geq a] = \sum_{x \in \Omega : X(x) \geq a} \mathbf{Pr}[x]$$

For ex. in the previous example, for the event $X = 50$ we have $\mathbf{Pr}[X = 50] = \frac{\binom{100}{50}}{2^{100}}$ (*)

$\rightarrow$ Given an event A define the indicator r.v. I_A:

$$I_A = \begin{cases} 1 & \text{if } A \text{ true} \\ 0 & \text{otherwise} \end{cases}$$

For ex. if $A =$ the event of having exactly 50 wins, $\mathbf{Pr}[A] = \mathbf{Pr}[I_A = 1]$, which is exactly (*)

Expectation

The expectation of a r.v. $X : \Omega \to \mathbb{R}$ *EXPX* is defined as

$$\mathbf{E}[X] = \sum_{x \in X(\Omega)} x \cdot \mathbf{Pr}[X = x].$$

Expectation (mean, average) is just the weighted sum over all values of the r.v.

Notice: If X is a r.v. then $\mathbf{E}[X] \in \mathbb{R}$.

Let X be an integer generated u.a.r. between 1 and 6. Then $\mathbf{E}[X] = \sum_{x=1}^{6} x \cdot \mathbf{Pr}[X = x] = \sum_{x=1}^{6} \frac{x}{6} = 3.5$, which is not a possible value for X.

Third trick: Linearity of expectation

So far we have seeing 2 important useful tricks:

1. Union Bound
2. Indicator random variables

Theorem

1. Given r.v. X, Y, $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.
2. Given $c \in \Theta(1)$, and a rv X, then $\mathbf{E}[cX] = c\mathbf{E}[X]$.
3. Given r.v. $\{X_i\}_{i=1}^n$, $\mathbf{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbf{E}[X_i]$.

The proof is standard and rely in the fact that the sum of r.v. is a r.v.

Independent r.v.

Two random variables X and Y are said to be independent if

$$\forall x, y \in \mathbb{R}, \mathbf{Pr}\left[(X = x) \cap (Y = y)\right] = \mathbf{Pr}\left[X = x\right] \cdot \mathbf{Pr}\left[Y = y\right].$$

Two not independent r.v. are said to be correlated.

Rolling two dices, let X_1 be a r.v. counting the pips in dice 1, and let X_2 be a r.v. counting the pips in dice 2. Then X_1 and X_2 are independents.

Rolling two dices, let X_1 be a r.v. counting the pips in dice 1, and let X_3 count the sum of pips in the two rollings, then X_1 and X_3 are correlated.

Interesting Example

Given and array $A[1, \dots, n]$ containing n different keys, chosen u.a.r. from one permutation of the set of n keys, let a_i, $1 \leq i \leq n$, be the key contained in $A[i]$. We say a_i and a_j are inverted if $i < j$ but $a_i > a_j$. Compute the expected number of inversions in A.

Let X count the number of inversions in A.

For every pair $1 \leq i < j \leq n$ of positions in A define an indicator r.v.:

$$X_{i,j} = \begin{cases} 1 & \text{if } a_i > a_j \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{i<j} X_{i,j} \Rightarrow \mathbf{E}[X] = \sum_{i<j} \mathbf{E}[X_{i,j}] = \sum_{i<j} 1 \cdot \underbrace{\mathbf{Pr}[a_i > a_j]}_{=1/2}$$

Notice $|\{(i,j) | 1 \leq i < j \leq n\}| = (n-1) + (n-2) + \cdots + 2 + 1$

therefore, $\mathbf{E}[X] = \frac{1}{2} \sum_{i=1}^{n} (n-i) = \frac{1}{2} \sum_{i=1}^{n-1} i = \frac{n(n-1)}{4}$

Deterministic algorithm to hire a student

We have n students $\{1, \ldots, n\}$, we want to hire the best one to help us. For that we have to interview one by one, each time we find one that is more suitable that the previous ones, we preselect him. At the end we hire the last one pre-selected, but we indemnify with $S > 0$ €, each of the pre-selected not hired. We want to minimize the number of students pre-selected.

Hiring (n)
best=0
for $i = 1$ to n **do**
  interview i
  if i is better than best **then**
    best=i and pre-select i

Adversarial complexity

The adversary gives you a list of ordered students s.t. you are forced to pre-select each of them.

$$T(n) = \Theta(n).$$

Average analysis of the hiring algorithm

The number of all possible order of the students is $n!$
We select u.a.r. an order with probability $= \frac{1}{n!}$.

Lemma The expected number of pre-selected is $O(\lg n)$
Proof Let X be a r.v. counting the number of pre-selected students.
For each $1 \leq i \leq n$ definean indicator r.v.

$$X_i = \begin{cases} 1 & \text{if } i \text{ is-pre-selected;} \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$X = \sum_{i=1}^{n} X_i \Rightarrow \mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} 1 \cdot \underbrace{\frac{1}{i}}_{\text{why?}} = O(\ln n). \square$$

Randomized algorithm for the hiring an student problem

To fool the input given by the adversary: Permute the input

Rand-Hire-Student (n)
Randomly permute the list $[n]$
best:=0
for $i = 1$ to n **do**
  interview i
  if i is better than best **then**
    best:=i and pre-select i

Let $X(n)$ a r.v. counting the number of pre-selections, on an input of n students. Then

$$\mathbf{E}\left[X(n)\right] = O(\ln n)$$