Reductions and Completeness

AA FIB, UPC

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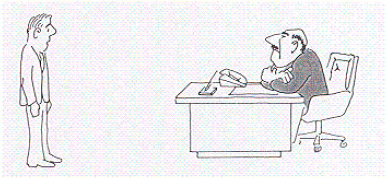
P vs NP

The US$ 10^6 Question: Is P $\neq$ NP or P $=$ NP ?

http://www.claymath.org/prizeproblems/pvsnp.html

Cartoons from Garey-Johnson's book



"I can't find an efficient algorithm,
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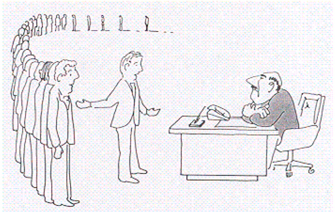

"I can't find an efficient algorithm,
but neither can all these famous people."

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- *Problem A is* log space many-one reducible *to problem B* ($A \leq_m^L B$) if there is a function f computable in logarithmic space so that $x \in L_A$ iff $f(x) \in L_B$.

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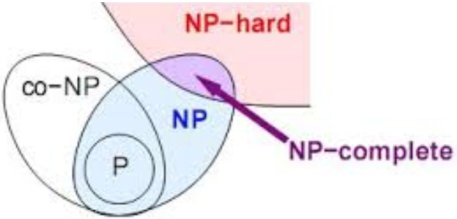
- A NP-complete problem is a problem complete for NP under $\leq_m^p$.
- $\leq_m^p$ and $\leq_m^L$ are transitive relations.

NP-completeness

A problem A is NP-complete if:

1. $A \in$ NP, and
2. for every $B \in$ NP, $B \leq_m^p A$.

If for each $B \in$ NP, $B \leq_m^p A$ but $A \notin$ NP then A is said to be NP-hard.



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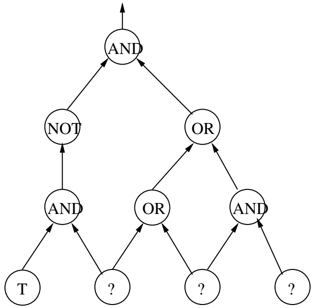
- *Majority conjecture:* P $\neq$ NP
- To prove a problem is NP-complete, we just have to find a reduction from a problem known to be NP-complete.
- We need as a seed a first NP-complete problem.

CIRCUIT SAT.

CIRCUIT SAT: Given a Boolean circuit with gates *AND*, *OR*, *NOT*, and the input gates *T*, *F* and *?*, and one output gate. Is there an an assignment to the input gates (?), such that the circuit evaluates to T?



For example if the input to ? is T,F,T, the output is F
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- There is a way to feed c and get output 1 iff there is a valid certificate.

The seminal theorem: Cook-Levin's Theorem

Therefore, given any instance x for A, we can construct in poly-time instance C of CIRCUIT SAT whose known inputs are the bits of x, and whose unknown inputs are the bits of c, and such that the output of C is 1 iff $\mathcal{A}$ outputs YES on input (x, c).

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Theorem (Cook-Levin's theorem)

CIRCUIT-SAT is NP-complete.

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is satisfiable, take $T(x_1) = T(x_2) = 1$, $T(x_3) = 0$ then $T(\phi) = 0$.

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k-SAT is NP-complete for $k \geq 3$ and in P for $k \leq 2$.

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- $SAT = \{F \mid \exists \sigma \in \{0,1\}^n F(\sigma) = 1\}$

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- By construction, C is SAT iff F is SAT. Besides, each gate is replaced by a constant number of clauses, so the construction requires polynomial time.

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- The last trick allow us to reduce k-SAT to $k+1$-SAT in polynomial time.

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Theorem

Clique, Independent set and Vertex cover are NP-complete.

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