A glimpse into Algorithmic Game Theory:

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Basic References non-coop game theory

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- Nisan et al. Eds. Algorithmic game theory, Cambridge University Press, 2007
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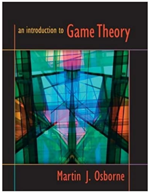
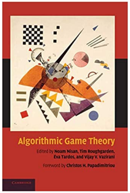
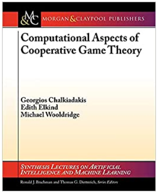






Where to use game theory?

Game theory studies decisions made in an environment in which players interact. The choice of optimal behavior when personal/common costs and benefits depend upon the choices of all participants assuming rational behaviour.

What for?

Game theory looks for states of equilibrium sometimes calles solutions and analyzes interpretations/properties of such states

Types of games

- Non-cooperative games
  - strategic games
  - extensive games
  - repeated games
  - Bayesian games
- Cooperative games
  - simple games
  - transferable utility games
  - non-transferable utility games
  - ...

One example: Strategic games

The story

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1 1eur, otherwise person 1 pays person 2 1eur.
- Payoff are equal to the amounts of money involved.

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Rationality: Players choose independently actions in order to maximize personal utility (minimize cost)

Example: Prisoner's Dilemma

The story

- Two suspects in a major crime are held in separate cells.
- Evidence to convict each of them of a minor crime.
- No evidence to convict either of them of a major crime unless one of them finks.

The penalties

- If both stay quiet, be convicted for a minor offense (1 year).
- If only one finks, he will be freed (and used as a witness) and the other will be convicted for a major offense (4 years).
- If both fink, each one will be convicted for a major offense with a reward for coperation (3 years each).

Prisoner's Dilemma

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

cost	Quiet	Fink
Quiet	1,1	3,0
Fink	0,3	2,2

The Prisoner's Dilemma models a situation in which

- there is a gain from cooperation,
- but each player has an incentive to free ride.

Strategic game

A strategic game Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \ldots, n\}$ of players.
- For each player $i \in N$, a nonempty set of actions A_i.
- Each player chooses his action once. Players choose actions simultaneously.
  No player is informed, when he chooses his action, of the actions chosen by others.
- For each player $i \in N$, a utility/cost function on the set of strategy profiles $A = A_1 \times \cdots \times A_n$.

Example: Sending from s to t

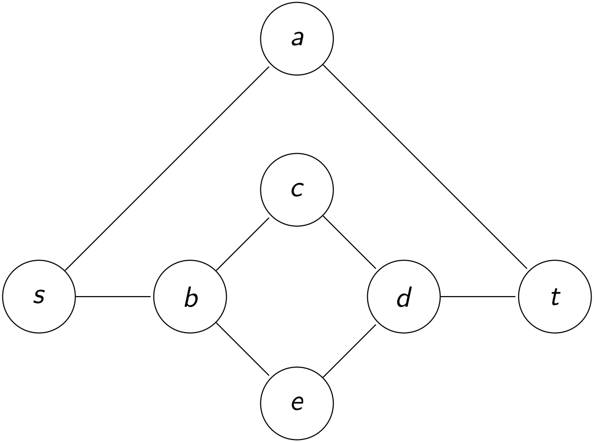
The story

- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \dots, v_{n-1})$.
- Pay-offs are defined as follows:
  player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \dots, v_{n-1}$ contains (u, v_u), otherwise gets 0.

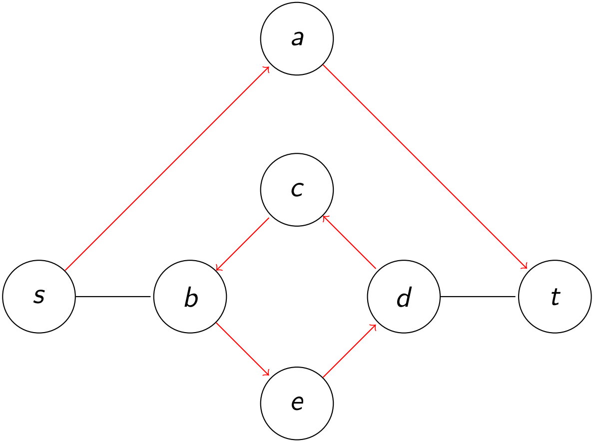
Players are selfish but the system can get a different reward/cost. For example the cost of the shortest path.

Usually a natural social cost/utility is involved in the model.

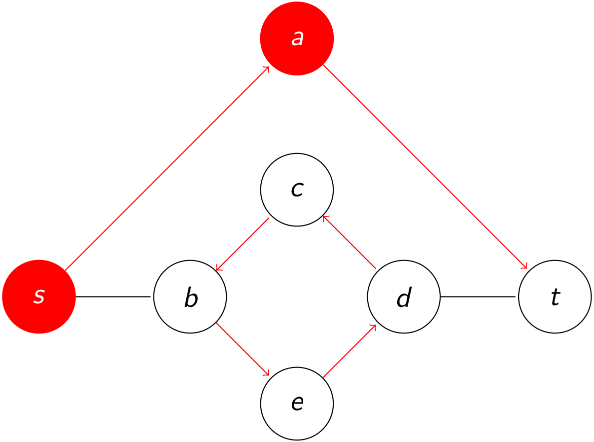
Sending from s to t: example



Sending from s to t: strategy profile (1)

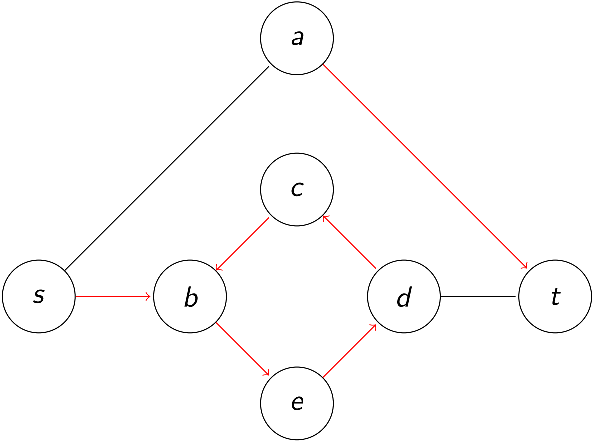


Sending from s to t: pay-offs (1)



Red nodes get pay-off 1, other nodes get pay-off 0.

Sending from s to t: strategy profile (2)



All nodes get pay-off 0.

Strategies: Notation

A strategy of player $i \in N$ in a strategic game Γ is an action $a_i \in A_i$.
A strategy profile $s = (s_1, \ldots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \ldots s_n)$ and $s'_i \in A_i$ we denote by

$(s_{-i}, s'_i) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
is not an strategy profile but can be seen as an strategy for the other players.

Best response and PNE

Let Γ be an strategic game defined through pay-off functions.
For $s = (s_1, \dots s_n)$ and $s'_i \in A_i$, let $(s_{-i}, s'_i) = (s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$

- The set of best responses for player i to strategy profile s is

$$BR_i(s_{-i}) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a'_i \in A_i} u_i(s_{-i}, a'_i)\}$$

- A pure Nash equilibrium is an strategy profile $s^* = (s^*_1, \dots, s^*_n)$ such that, for each player i, $s^*_i \in BR_i(s^*)$.

Pure Nash Equilibrium

- Is a strategy profile in which all players are happy.
- Identified with a fixed point of an iterative process of computing a best response.
- However, the game is played only once!
- GT deals with the existence and analysis of equilibria assuming rational behavior.
  players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

More games and their PNE

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

utility	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

utility	swerve	don't sw
swerve	3,3	2,4
don't sw	4,2	1,1

- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Matching Pennies, none.
- Chicken, (swerve, don't sw), (don't sw, swerve).

Pure Nash equilibrium

- First notion of equilibrium for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?

When considering probability distributions on the set of actions as strategies. Setting individual utility as expected utility under the product distribution, we can redefine Best Response and Nash equilibrium notions. For such type of strategies every strategic game has a Nash equilibrium.

Game Theory and CS

- Framework to analyze equilibrium states of protocols used by rational agents.
  Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
  Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
  Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
  Algorithmic game theory

Characteristic Function Games

- A characteristic function game is a pair (N, v), where:
  - $N = \{1, ..., n\}$ is the set of players and
  - $v : \mathcal{C}_N \to \mathbb{R}$ is the characteristic function.
  - for each coalition of players $C \subseteq N$, $v(C)$ is the amount that the members of C can earn by working together
- usually it is assumed that v is
  - normalized: $v(\emptyset) = 0$,
  - non-negative: $v(C) \geq 0$, for any $C \subseteq N$, and
  - monotone: $v(C) \leq v(D)$, for any C, D such that $C \subseteq D$

One example: Cooperative games

- We have a group of n children, each has some amount of money the i-th child has b_i dollars.
- There are three types of ice-cream tubs for sale:

  Type 1 costs \$7, contains 500g
  Type 2 costs \$9, contains 750g
  Type 3 costs \$11, contains 1kg

- The children have utility for ice-cream but do not care about money.
- The payoff of each group is the maximum quantity of ice-cream the members of the group can buy by pooling all their money.
- The ice-cream can be shared arbitrarily within the group.

Ice-Cream Game: Characteristic Function

Charlie: \$6

Marcie: \$4

Pattie: \$3

$w = 500$

$p = \$7$

$w = 750$

$p = \$9$

$w = 100$

$p = \$11$

- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 750, v(\{C, P\}) = 750, v(\{M, P\}) = 500$
- $v(\{C, M, P\}) = 1000$

Solution concepts

- An outcome of a game $\Gamma = (N, v)$ is a payoff vector $x = (x_1, ..., x_n)$, which distributes the value of the grand coalition N.
- An outcome (P, x) is called an imputation if it satisfies individual rationality: $x_i \geq v(\{i\})$, for $i \in N$.

Outcome:example

Suppose $v(\{1,2,3\}) = 9$ and $v(\{4,5\}) = 4$

- $((\{1,2,3\},\{4,5\}),(3,3,3,3,1))$ is an outcome
- $((\{1,2,3\},\{4,5\}),(2,3,2,3,3))$ is NOT an outcome as transfers between coalitions are not allowed

What Is a Good Outcome?

 Charlie: \$4  Marcie: \$3  Pattie: \$3

Ice-cream pots: $w = (500, 750, 100)$ and $p = (\$7, \$9, \$11)$

- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 500, v(\{C, P\}) = 500, v(\{M, P\}) = 0$
- $v(\{C, M, P\}) = 750$

This is a superadditive game, so outcomes are payoff vectors!

How should the players share the ice-cream?

- $(200, 200, 350)$ $v(\{C, M\}) > x(\{C, M\})$, not stable!
- $(250, 250, 250)$ alone or in pairs do not get more, stable!.
- (750, 0, 0) is also stable!

The core

The core of a game Γ is the set of all stable outcomes, i.e., outcomes that no coalition wants to deviate from

$$\text{core}(\Gamma) = \{(P, x) | x(C) \geq v(C) \text{ for any } C \subseteq N\}$$

each coalition earns, according to x, at least as much as it can make on its own.

- no subgroup of players can deviate so that each member of the subgroup gets more.
- Main computational problem: Is the core empty? if not, compute elements in it.

Stability vs. Fairness

- Outcomes in the core may be unfair.
- A fair payment scheme rewards agents according to their contribution.
- Attempt: given a game $\Gamma = (N, v)$, set

$$x_i = v(\{1, ..., i-1, i\}) - v(\{1, ..., i-1\}).$$

  The payoff to each player is his marginal contribution to the coalition of his predecessors
- Idea: Remove the dependence on ordering taking the average over all possible orderings.

Shapley Value

- A permutation of $\{1, ..., n\}$ is a one-to-one mapping from $\{1, ..., n\}$ to itself
  $\Pi(N)$ denotes the set of all permutations of N
- Let $S_\pi(i)$ denote the set of predecessors of i in $\pi \in \Pi(N)$
- For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$
- The Shapley value of player i in a game $\Gamma = (N, v)$ with n players is

$$\Phi_i(\Gamma) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \delta_i(S_\pi(i))$$

- Φ_i is i's average marginal contribution to the coalition of its predecessors, over all permutations

What Is a Good Outcome?

 Charlie: \$4  Marcie: \$3  Pattie: \$3

Ice-cream pots: $w = (500, 750, 100)$ and $p = (\$7, \$9, \$11)$

- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 500, v(\{C, P\}) = 500, v(\{M, P\}) = 0$
- $v(\{C, M, P\}) = 750$

	C	M	P
CMP	0	500	250
CPM	0	250	500
MCP	500	0	250
MPC	750	0	0
PCM	500	250	0
PMC	750	0	0

Shapley values are $(2500/6, 1000/6, 1000/6)$

Shapley Value: Probabilistic Interpretation

- Φ_i is i's average marginal contribution to the coalition of its predecessors, over all permutations
- Suppose that we choose a permutation of players uniformly at random, then Φ_i is the expected marginal contribution of player i to the coalition of his predecessors

Game Theory and CS

- Framework to analyze cooperative behaviour.
  Modeling.
- New concepts to analyze/justify individual relevance: Power indices.
  Give criteria to select/address/control relevant participants
- Basic tools to study basic decision processes in particular elections
  Computational Social Choice
- Source of new computational problems to study.
  Algorithmic game theory