

Parameterized algorithms: Tree width and dynamic programming

Maria Serna

Spring 2025-2026

- 1 Tree width
- 2 Nice tree decomposition
- 3 Min Vertex cover parameterized by treewidth

Graph parameters

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- We are going to explore a parameter that measures the closeness of a graph to a tree: **treewidth**.
- A similar parameter measures closeness of a graph to a path: **pathwidth**.

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- For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v .
 $N[v] = N(v) + v$. $d(v) = |N(v)|$.
- For a graph $G = (V, E)$, $\delta(G) = \min_{v \in V} d(v)$, and
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- A graph is **outerplanar** if it can be drawn as a cycle with non-crossing chords.

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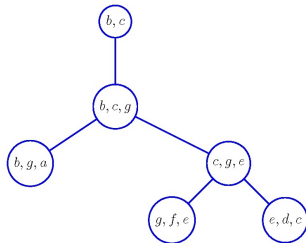
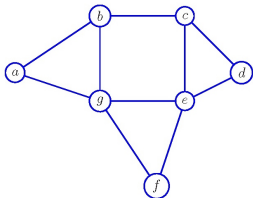
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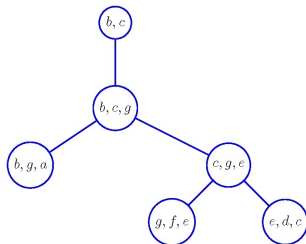
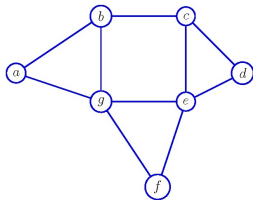
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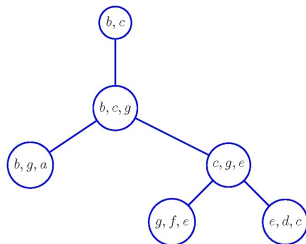
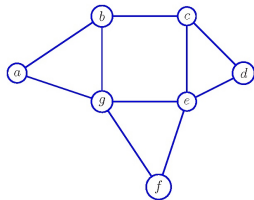
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- Equivalently the second condition can be expressed as:
 - For every $u, v \in V(T)$ and every vertex w on the path between u and v , $X_u \cap X_v \subseteq X_w$, and every vertex of G appears in at least one X_v .

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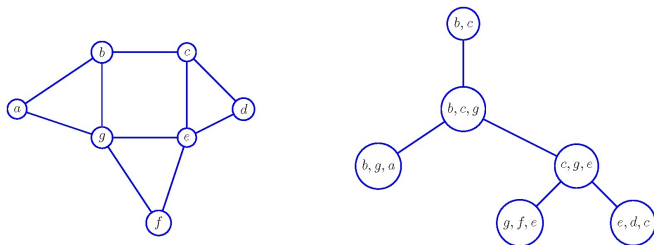


Tree decomposition



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- The sets X_v are called the **bags** of the tree decomposition.

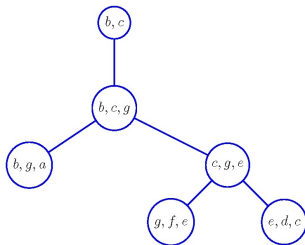
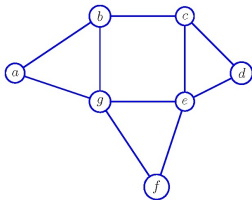
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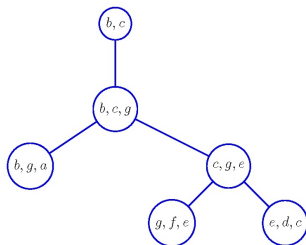
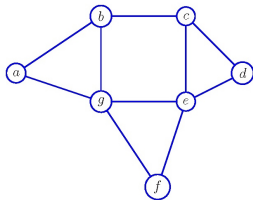
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- The **tree width** ($tw(G)$) of a graph G is the minimum width over all tree decompositions of G .

A graph with tree width 2



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This graph is an **outerplanar** graph.

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- For K_n , the complete graph on n vertices, $tw(K_n) = n - 1$.

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- There is a polynomial time approximation algorithm that given a (G, k) decides in polynomial time whether $tw(G) \leq k$ and if so produces a tree decomposition of width $\leq r \cdot k$ and size $O(|V|)$, for some constant $r > 1$.

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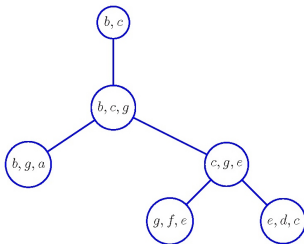
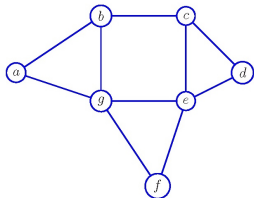
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Thus, we have an induced subgraph associated to each node.
- Notice that X_v is a separator in G .



Exercise For this rooted tree decomposition. Draw the graphs associated to each node in the tree. What are the differences among parent-child graphs?

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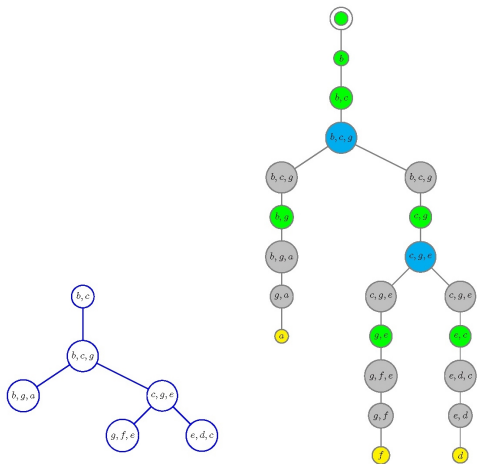
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 - with $X_v \subseteq X_u$ and $|X_u| = |X_v| + 1$ (introduce)

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 - u has two children v and w with $X_u = X_v = X_w$ (join)

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Nice tree decomposition

Lemma

Computing a rooted nice tree decomposition with width at most k , given a small tree decomposition of width at most k takes $O(kn)$ time.

Nice tree decomposition

- Nodes in the tree
node u holds a subset of vertices X_u , and has a subgraph G_u associated to it.
- the root r has $X_r = \emptyset$ and $G_r = G$.
- nodes can be of four types:

Start

Introduce

Forget

Join

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A parameterization for Vertex Cover?

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Input: A graph G and an integer k ,

Parameter: $tw(G) + k$

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PROBLEM: $tw(G)$ cannot be computed in polynomial time (unless $P=NP$)!

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and an integer k ,

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TW-K-VERTEX COLORING

Input: A graph G and integers w and k ,

Parameter: $w + k$

Question: Does G have a vertex cover with k vertices and $tw(G) \leq w$?

The same kind of tw parameterization applies to other graph problems.

For some graph properties, an upper bound in terms of treewidth can be found. For such problems the parameter might be only w .

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- We deal with each type of node separately

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The associated graph has just a node, therefore the values are correct.

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All neighbors of x in $G(u)$ are in X_v and therefore in C since C is a vertex cover of $G[X_u]$.

Forget node

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Let u be a *forget* node, let v be its unique child and assume that $\{v\} = X_v - X_u$.

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Let u be a *forget* node, let v be its unique child and assume that

$$\{v\} = X_v - X_u.$$

Then for all $C \subseteq X_u$, $s_u(C) = \min\{s_v(C), s_v(C + x)\}$.

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 - If $x \in C'$ then similarly $|C'| \geq s_v(C + x)$.
- (\leq) Let C_1 and C_2 be the VCs that determine $s_v(C)$ and $s_v(C + x)$ respectively. C_1, C_2 are VC of $G(u)$ compatible with C , so $s_v(C) \leq \min\{|C_1|, |C_2|\}$.



Join node

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Claim

Let u be a join node of T with children v and w .

Then for all $C \subseteq X_u$: $s_u(C) = s_v(C) + s_w(C) - |C|$.

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- (\geq) If C' is a vertex cover of $G(u)$ with $C' \cap X_u = C$, then $C' \cap V(v)$ is a vertex cover of $G(v)$ and $C' \cap V(w)$ is a vertex cover of $G(w)$. These sets share $|C|$ vertices.

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- (\leq) Two C -compatible vertex covers of $G(v)$ and $G(w)$ of size $s_v(C)$ and $s_w(C)$ can be combined to a vertex cover of $G(u)$ of size $s_v(C) + s_w(C) - |C|$.



Min Vertex cover parameterized by treewidth

Theorem

Let (T, X) be a rooted nice tree decomposition of width w of a graph G on n vertices. In time $2^{w+1}n^{O(1)}$ the size of a minimum vertex cover of G can be computed.

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- We can construct a minimum vertex cover as well, by tracing back through the tree decomposition.
- + $O(f(k)n)$ to get the tree decomposition if needed.

