Linear Programming approximation: Primal Dual

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Primal Dual

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- Relax+Round: solve the linear program in polynomial time, and round the solution.
- However, for certain problems, we do not even need to solve the LP to get good (reasonable approximation factor or even optimal) solutions to our problem using duality to control improvements.

Historical notes

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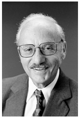
- George Dantzig started linear programming (1947) , and his ideas contain the first germs of primal dual algorithms. The Hungarian method was an application of the paradigm.
- Jack R. Edmonds gave the first (sophisticated) application of the paradigm in his work on maximum weight matchings in arbitrary graphs (1965).
- Bar-Yehuda and Even first enunciated the paradigm in their work on the weighted Vertex Cover problem (1981).

Dantzig  Edmonds 

Primal, Dual and Weak Duality

Consider a LP in n variables $x = (x_1, \ldots, x_n)$ with m constraints represented by matrix A, independent terms b, and objective function b.

Primal

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The dual is an effort to construct the best lower bound for the primal objective function.

Searching for a lower bound: The best one?

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DUAL

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- Working from the dual trying to get the best lower bound we come back to the primal.
- Another example that you know is the pair MaxFlow-MinCut if you write the LP formulation of MaxFlow you can check that the dual is a LP formulation for MinCut

Strong and Weak duality theorem

There are additional conditions for a pair (x, y) of primal-dual optimal/feasible solutions.

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If the primal has an optimal solution x^* then the dual has an optimal solution y^* and $c^T x^* = b^T y^*$

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Theorem (Weak Duality)

For every feasible solution x to the primal and every feasible solution z to the dual,

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j z_j$$

Conditions for optimality: Complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal complementary slackness

If $x_i > 0$, then $\sum_{j=1}^{m} a_{ij} z_j = c_i$.

Dual complementary slackness

If $z_j > 0$, then $\sum_{i=1}^{n} a_{ij} x_i = b_j$.

Conditions for optimality: Complementary slackness

Theorem

If (x, y) satisfies primal and dual complementary slackness, then x and y are optimal solutions for primal and dual problems, respectively.

Proof.

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij} z_j) x_i = \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij} x_i) z_j = \sum_{j=1}^{m} b_j z_j$$

☺

Relaxed complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal relaxed complementary slackness

If $x_i > 0$, then $\sum_{j=1}^{m} a_{ij} z_j \geq c_i/\alpha$.

Dual relaxed complementary slackness

If $z_j > 0$, then $\sum_{i=1}^{n} a_{ij} x_i \leq \beta\, b_j$.

for some factors $\alpha, \beta \geq 1$

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If x is integral and primal and dual relaxed complementary slackness hold?

Relaxed complementary slackness

Theorem

Let Π be a minimization integer program and Π-LP its LP-relaxation. Suppose a primal (integer) feasible solution x of Π and a dual feasible solution y of Π-LP satisfy the primal-dual relaxed complementary slackness, for some $\alpha, \beta > 1$*, and x is integral, then x is a* $\alpha\beta$*-approximation.*

Relaxed complementary slackness

Proof.

$$\sum_{i=1}^{n} c_i x_i \leq \alpha \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij} z_j) x_i = \alpha \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij} x_i) z_j \leq \alpha\beta \sum_{j=1}^{m} b_j z_j$$

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$$\sum_{i=1}^{n} c_i x_i \leq \alpha\beta \sum_{j=1}^{m} b_j z_j \leq \alpha\beta \, \text{opt}$$

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- If at some point objective functions match, we have found an optimal solution.
- If at some point relaxed complementary slackness holds, for some r, we have found a r-approximate solution.

Primal-Dual vertex cover formulation

VC

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- We know how to relax the IP formulation as LP problem

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- We know how to relax the IP formulation as LP problem
- We know how to compute the dual of the LP problem

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

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Primal-Dual approximation: relaxed complementary conditions

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- So, relaxed complementary slackness conditions hold for $r = 2$. A 2-approximation ☺.

Primal-Dual for weighted vertex cover

WVC

Given a vertex weighted graph $G = (V, A, c)$ we want to find a set $S \subset V$ with minimum weight, so that every edge in G has at least one end point in S.

- The problem is NP-hard and belongs to NPO.
- Can we formulate WVC as an IP problem?

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- Restrictions: for every edge $(i, j) \in E$, $x_i + x_j \geq 1$
- $x_i \in \{0, 1\}$
- The IP can be computed in polytime.

Weighted vertex cover: LP relaxation

IP

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- Cost of the solution? x, z are feasible. Relaxed complementary slackness conditions?

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- So, relaxed complementary conditions hold for $r = 2$ and we have a 2-approximation for WVC.

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  In general, this step can be implemented solving another LP program!
- We can also increase edge variables one by one. This leads to another primal-dual approximation algorithm PRICING METHOD

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- A vertex is tight with respect to a pricing z if $\sum_{i \in e} z_e = c_i$.

Pricing algorithm

Set prices and find vertex cover simultaneously.

function PRICING WVC(G, c)
 $S = \emptyset$;
 for $e \in E$ **do**
  $z[e] = 0$ % initial price is 0
 while there is $(i, j) \in E$ so that neither i nor j is tight **do**
  select such an edge $e = (i, j)$
  Increase $z[e]$ until i or j became tight.
  Add to S the vertex (vertices) that became tight.
 return S

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- However, PRICING WVC is a greedy algorithm.
- No LP solver has been used!

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- The rounding step should pay a factor at least equal to the integrality gap of the relaxation

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- It is not possible to prove better than 2 approximation for VC with LP VC ☹

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  Thus any r-approximating algorithm for Max Clique can be used to get a r-approximation algorithm for Max IS (and viceversa)

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  This reduction does not preserve approximability ☹