

Linear Programming approximation: Primal Dual

Maria Serna

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- 1 LP duality
- 2 Approximating Vertex Cover
- 3 Approximability limits

Primal Dual

- Relax+Round: solve the linear program in polynomial time, and round the solution.
- However, for certain problems, we do not even need to solve the LP to get good (reasonable approximation factor or even optimal) solutions to our problem using duality to control improvements.

Historical notes

- George Dantzig started linear programming (1947) , and his ideas contain the first germs of primal dual algorithms. The Hungarian method was an application of the paradigm.
- Jack R. Edmonds gave the first (sophisticated) application of the paradigm in his work on maximum weight matchings in arbitrary graphs (1965).
- Bar-Yehuda and Even first enunciated the paradigm in their work on the weighted Vertex Cover problem (1981).



Dantzig



Edmonds

Primal, Dual and Weak Duality

Consider a LP in n variables $x = (x_1, \dots, x_n)$ with m constraints represented by matrix A , independent terms b , and objective function b .

Primal

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

The **dual** is an effort to construct the best lower bound for the primal objective function.

Searching for a lower bound: The best one?

LP (PRIMAL)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

if x^* opt, $y^T Ax$ is a general linear combination of equations, if we can select y so that $y^T Ax^* = c^T x^*$,
 $c^T x^* \geq y^T b$

The best lower bound, for any x ?

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y = c \\ & y \geq 0 \end{aligned}$$

But as we are maximizing this is equivalent to

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \quad \text{DUAL} \\ & y \geq 0 \end{aligned}$$

Primal - Dual: an example

Primal - Dual: an example

- Working from the dual trying to get the best lower bound we come back to the primal.
- Another example that you know is the pair MaxFlow-MinCut if you write the LP formulation of MaxFlow you can check that the dual is a LP formulation for MinCut

Strong and Weak duality theorem

There are additional conditions for a pair (x, y) of primal-dual optimal/feasible solutions.

Theorem (Strong duality)

If the primal has an optimal solution x^ then the dual has an optimal solution y^* and $c^T x^* = b^T y^*$*

Theorem (Weak Duality)

For every feasible solution x to the primal and every feasible solution z to the dual,

$$\sum_{i=1}^n c_i x_i \geq \sum_{j=1}^m b_j z_j$$

Conditions for optimality: Complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal complementary slackness

If $x_i > 0$, then $\sum_{j=1}^m a_{ij}z_j = c_i$.

Dual complementary slackness

If $z_j > 0$, then $\sum_{i=1}^n a_{ij}x_i = b_j$.

Conditions for optimality: Complementary slackness

Theorem

If (x, y) satisfies primal and dual complementary slackness, then x and y are optimal solutions for primal and dual problems, respectively.

Proof.

$$\sum_{i=1}^n c_i x_i = \sum_{i=1}^n \left(\sum_{j=1}^m a_{ij} z_j \right) x_i = \sum_{j=1}^m \left(\sum_{i=1}^n a_{ij} x_i \right) z_j = \sum_{j=1}^m b_j z_j$$



Relaxed complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal relaxed complementary slackness

If $x_i > 0$, then $\sum_{j=1}^m a_{ij}z_j \geq c_i/\alpha$.

Dual relaxed complementary slackness

If $z_j > 0$, then $\sum_{i=1}^n a_{ij}x_i \leq \beta b_j$.

for some factors $\alpha, \beta \geq 1$

If x is integral and primal and dual relaxed complementary slackness hold?

Relaxed complementary slackness

Theorem

Let Π be a minimization integer program and Π -LP its LP-relaxation. Suppose a primal (integer) feasible solution x of Π and a dual feasible solution y of Π -LP satisfy the primal-dual relaxed complementary slackness, for some $\alpha, \beta > 1$, and x is integral, then x is a $\alpha\beta$ -approximation.

Relaxed complementary slackness

Proof.

$$\sum_{i=1}^n c_i x_i \leq \alpha \sum_{i=1}^n \left(\sum_{j=1}^m a_{ij} z_j \right) x_i = \alpha \sum_{j=1}^m \left(\sum_{i=1}^n a_{ij} x_i \right) z_j \leq \alpha \beta \sum_{j=1}^m b_j z_j$$

By weak duality $\sum_{j=1}^m b_j z_j \leq \sum_{i=1}^n c_i x'_i$ for any feasible x' , in particular for the optimal solution of the IP, therefore

$$\sum_{i=1}^n c_i x_i \leq \alpha \beta \sum_{j=1}^m b_j z_j \leq \alpha \beta \text{opt}$$



Primal dual algorithms

- Primal-Dual algorithms iterate obtaining primal/dual feasible solutions by increasing values of variables until a restriction is **tight** (fulfilled with equality).
- If at some point objective functions match, we have found an optimal solution.
- If at some point relaxed complementary slackness holds, for some r , we have found a r -approximate solution.

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Primal-Dual vertex cover formulation

VC

Given a graph $G = (V, E)$, we want to find a set S , with minimum number of vertices, so that every edge in G has at least one end point in S .

- We know how to formulate VC as an IP problem
- We know how to relax the IP formulation as LP problem
- We know how to compute the dual of the LP problem

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^n x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \in \{0, 1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^n x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{aligned}$$

- Let opt be the size of an optimal solution of the VC instance.
- Let x^* be an optimal solution of the LP and $s^* = \sum_{i=1}^n x_i^*$.
- $s^* \leq \text{opt}$

Vertex cover: Primal-Dual approximation

LP primal

$$\begin{aligned} \min \quad & \sum_{i=1}^n x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad e = (i, j) \in E \\ & x_i \geq 0 \quad i \in V \end{aligned}$$

LP dual

$$\begin{aligned} \max \quad & \sum_{e \in E} z_e \\ \text{s.t.} \quad & \sum_{i \in e} z_e \leq 1 \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{aligned}$$

- Start with the integer infeasible primal solution $x = 0$, and the dual feasible solution $z = 0$.
- Repeat while some constraint in primal is unsatisfied:
 - Increase all (unfrozen) variables z_e until some dual constraint becomes tight (say, for vertex i).
 - For each dual tight constraint, $x_i = 1$. Freeze all the variables z_e such that $i \in e$.

Vertex cover: Primal-Dual approximation

- When the process stops, we have increased the variables z_e suitably.
- Some vertices i were chosen ($x_i = 1$)
- This set S of vertices is our output.
- Is S a vertex cover?
Otherwise, we would have continued as some primal constraint were still unsatisfied.
- Cost of the solution?
At the end of the algorithm x, z are feasible. Relaxed complementary slackness?.

Primal-Dual approximation: relaxed complementary conditions

Two conditions hold:

- Primal:

If $x_i > 0$, we have frozen $x_i = 1$ at some step, then $\sum_{i \in e} z_e = 1$.

- Dual:

If $z_e > 0$, z_e is increased. We do not know if it is because for one or both endpoints the constraints got tight, but $x_i + x_j \leq 2 \leq 2c_i$, for $e = (i, j)$.

- So, relaxed complementary slackness conditions hold for $r = 2$. A 2-approximation 😊.

Primal-Dual for weighted vertex cover

WVC

Given a vertex weighted graph $G = (V, A, c)$ we want to find a set $S \subset V$ with minimum weight, so that every edge in G has at least one end point in S .

- The problem is NP-hard and belongs to NPO.
- Can we formulate **WVC** as an IP problem?
- Variables: $x_1 \dots x_n$, $x_i = 1$ iff $i \in S$.
- Objective function: $\sum_{i=1}^n c_i x_i$.
- Restrictions: for every edge $(i, j) \in E$, $x_i + x_j \geq 1$
- $x_i \in \{0, 1\}$
- The IP can be computed in polytime.

Weighted vertex cover: LP relaxation

IP

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i x_i \\ \text{s.t.} & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \in \{0, 1\} \quad \text{for all } i \in V \end{array}$$

LP

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i x_i \\ \text{s.t.} & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{array}$$

Weighted vertex cover: Primal-Dual approximation

LP primal

$$\begin{aligned} \min \quad & \sum_{i=1}^n c_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{aligned}$$

LP dual

$$\begin{aligned} \max \quad & \sum_{e \in E} z_e \\ \text{s.t.} \quad & \sum_{i \in e} z_e \leq c_i \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{aligned}$$

- Start with the integer infeasible primal solution $x = 0$, and the dual feasible solution $z = 0$.
- Repeat while some constraint in primal is unsatisfied:
 - Increase all (unfrozen) variables z_e until some dual constraint becomes tight (say, for vertex i).
 - Set $x_i = 1$. Freeze all the variables z_e such that $i \in e$.

Weighted vertex cover: Primal-Dual approximation

- When the process stops, we have increased the variables z_e suitably.
- Some vertices i were chosen ($x_i = 1$)
- This set S of vertices is our output and again is a vertex cover.
- Cost of the solution? x, z are feasible. Relaxed complementary slackness conditions?

Primal-Dual approximation: relaxed complementary conditions

Two conditions hold:

- Primal:

If $x_i > 0$, we have frozen $x_i = 1$ at some step,
then $\sum_{i \in e} z_e = c_i$.

- Dual:

If $z_e > 0$, z_e is increased. We do not know if it is because for one or both endpoints the constraints got tight, but $x_i + x_j \leq 2 \leq 2c_i$, for $e = (i, j)$.

- So, relaxed complementary conditions hold for $r = 2$ and we have a 2-approximation for **WVC**.

Primal-Dual approximation: generalizing the approach

- In the algorithm, we increased the (active) dual variables simultaneously.
- Trying to get the highest (the best) lower bound that we can get for the primal minimization objective.
In general, this step can be implemented solving another LP program!
- We can also increase edge variables one by one. This leads to another primal-dual approximation algorithm **PRICING METHOD**

Pricing method: another view of Primal-Dual

- Each edge must be covered by some vertex.
- Edge $e = (i, j)$ pays price $z_e \geq 0$ to use both vertex i and j .
- Fairness: Edges incident to vertex i should pay $\leq c_i$ in total.
- Prices z_e are **fair** if, for any vertex cover S , $\sum_e z_e \leq w(S)$.
- A **vertex is tight** with respect to a pricing z if $\sum_{i \in e} z_e = c_i$.

Pricing algorithm

Set prices and find vertex cover simultaneously.

function PRICING WVC(G, c)

$S = \emptyset$;

for $e \in E$ **do**

$z[e] = 0$ % initial price is 0

while there is $(i, j) \in E$ so that neither i nor j is tight **do**

select such an edge $e = (i, j)$

Increase $z[e]$ until i or j became tight.

Add to S the vertex (vertices) that became tight.

return S

Pricing algorithm

Theorem

PRICING WVC is a 2-approximation for **WVC**.

- Follows directly from primal-dual arguments.
- However, **PRICING WVC** is a greedy algorithm.
- **No LP solver has been used!**

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Relax+Round: Integrality gap

- **Definition:** largest ratio on all instances between the optimum integral solution and the optimum relaxed solution.
- One cannot hope to achieve an approximation ratio better than the integrality gap of the relaxation
- The rounding step should pay a factor at least equal to the integrality gap of the relaxation

Integrality gap: VC

- On a clique graph the optimal vertex cover is of size $n - 1$
- $x^*(u) = 1/2$ is a feasible solution to LP VC of value $n/2$
- The integrality gap is equal to $2(1 - \frac{1}{n})$
- It is not possible to **prove better than 2 approximation** for VC with LP VC 😞

Showing Non-approximability: reductions

How to show non approximability, as always in Complexity translate results.

- From another approximability result via [Approximation Preserving \(AP\)-reductions](#)
Or other adequate reducibility notions
- From a NP-complete/hard problem via [gap-reductions](#)

Some classic reductions

- From **INDEPENDENT SET** to **CLIQUE**

$$f(G, k) = ((V, \{(u, v) \mid (u, v) \notin E\}), k)$$

- From **MAX IS** to **MAX CLIQUE?**

$$\text{Candidate: } f(G) = (V, \{(u, v) \mid (u, v) \notin E\})$$

Is there any relation about the respective optimum values?

They are the same!

Thus any r -approximating algorithm for **MAX CLIQUE** can be used to get a r -approximation algorithm for **MAX IS** (and viceversa)

Some classic reductions

- From **INDEPENDENT SET** to **VC**

$$f(G, k) = (G, n - k)$$

- From **MAX IS** to **MIN VC?**

Candidate: $f(G) = G$

Is there any relation about the respective optimum values?

$$\min VC(G) + \max IS(G) = |V|$$

One can be big and the other small so that a r -approximate solution for one problem will not define always a r -approximation for the corresponding instance.

This reduction does not preserve approximability 😞