Computational Complexity: Classes

AA-GEI FIB, UPC

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Problem types

- Decision
  Input x *in* I
  Is $P(x)$ *true?*

  Example: Given a graph and two vertices, is there a path joining them?

- Function
  Input x *in* I
  Compute y *such that* $Q(x, y)$

  Example: Given a graph and two vertices, compute the minimum distance between them.

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- We need to guarantee that $\{\langle x \rangle \mid x \in I\}$ is decidable.

Decision problem classes

- Undecidable
  No algorithm can solve the problem.
- Decidable
  There is an algorithm solving them.

Classifying decidable problems

- Decidable problems, admit algorithmic solutions.
- The algorithms solving such problems can use more or less amount of resources.
- To measure the performance of an algorithm we use two measures time and space.
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- As usual we take a worst case analysis on inputs with the same size.
- For a TM, time correspond to the length of the computation and space to the portion of the tape accessed during the computation.
- Observe that sometimes the space used might be smaller than the input.

Complexity classes: definitions

Let $f : \mathbb{N} \to \mathbb{N}$,

- The class TIME($f(n)$) is the class of decision problems for which an algorithm exists that solves instances of size n in time $O(f(n))$.
- The class SPACE($f(n)$) is the class of decision problems for which an algorithm exists that solves instances of size n using space $O(f(n))$.

Some complexity classes

- P $= \cup_k TIME(n^k) = TIME(poly(n))$
- EXP $= TIME(2^{poly(n)})$
- 2EXP $= TIME(2^{2^{poly(n)}})$
- L $=$ LOGSPACE $= SPACE(\lg n)$
- PSPACE $= SPACE(poly(n))$
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$$\mathsf{L} \subseteq \mathsf{P} \subseteq \mathsf{EXP} \qquad \mathsf{PSPACE} \subseteq \mathsf{EXP}$$

Non deterministic machines

- We can use a generalized, although unrealistic, model of computation.
- A program has a built-in operation allowing to branch to different positions in the code.
- When the code is executed, the machinefollows the the “best” continuation.

Complexity classes: definitions

Let $f : \mathbb{N} \to \mathbb{N}$,

- The class NTIME($f(n)$) is the class of decision problems that a non deterministic TM recognizes in time $O(f(n))$.
- The class NSPACE($f(n)$) is the class of decision problems for which a non deterministic TM solves instances of size n using space $O(f(n))$.

Some complexity classes

- NP $=$ NTIME$(poly(n))$
- NEXP $=$ NTIME$(2^{poly(n)})$
- NL $=$ NSPACE$(\lg n)$
- NPSPACE $=$ NSPACE$(poly(n))$

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$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXP}$$

Logic and NP

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- A polynomial time verifier runs in polynomial time in the length of x.
- A language L is polynomially verifiable if it has a polynomial time verifier.
- A verifier uses additional information, represented by the word c, to verify that a string x is a member of L. This information is called a certificate, or proof, of membership in L.
- A polynomial verifier can access only polynomial space. Therefore, we can assume that $|c|$ is polynomial in $|x|$.

Decision Problems in NP: Examples

Theorem

$L \in NP$ iff $L = \{x \mid \exists y\, R(x, y)\}$ where $|y| = poly(|x|)$ and R can be decided in polynomial time.

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NP is the class of polynomially verifiable languages.

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- SUBSET-SUM: Given a set $S = \{x_1, \ldots, x_n\}$ of $\mathbb{Z}^+$ and $t \in \mathbb{Z}$, exists $S' \subseteq S$ with $\sum_{x_j \in S'} x_j = t$?

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  - So, SUBSET-SUM $\in$ NP.
- HAMILTONIAN CYCLE: Given a graph G, is G Hamiltonian?
  - The certificate is a permutation of $V(G)$, it can be represented by a vector, so it has polynomial size. The verifier also runs in polynomial time.
  - So, HAMILTONIAN CYCLE $\in$ NP.

Classes of complements

- Let $\mathcal{C}$ be a class of decision problems. co-$\mathcal{C}$ is formed by the complements of languages in $\mathcal{C}$, i.e., $L \in$ co-$\mathcal{C}$ iff $\overline{L} \in \mathcal{C}$.

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- Classes defined by TMs are closed under complement.

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- Classes defined by TMs are closed under complement.

$$\text{P} = \text{co-P} \qquad \text{EXP} = \text{co-EXP} \qquad \text{PSPACE} = \text{co-PSPACE}$$

- Not so clear for classed defined by NTMs.
  co-NP $= \{x \mid \forall y R(x, y)\}$ where $|y| = poly(|x|)$ and R can be decided in polynomial time.
- It seems different to asses that a property holds for a leaf in a tree than on all their leaves.

Decision Problems in co-NP: Examples

- PRIME: Given an integer n, is n prime?
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  - PRIME $\in NP$ as it can be verified in poly time (Pratt 1975). So in NP $\cap$ co-NP
  - A polynomial time algorithm $O(\lg n^6)$ was devised by Agrawal, Kayal and Saxena in 2002.

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  - $p_1, \ldots, p_k$ is the certificate.
  - k and all the possible factors p are $\leq n$.
  - The verifier, on input n and $p_1, \ldots, p_k$, computes $p_1 \times \ldots \times p_k$, checks if the result is n. If so, checks that all the values are prime numbers.
  - This takes time $O(k \lg n^2 + k \lg n^6)$.
  - So, FACTORIZATION $\in$ NP.

Open questions

- Is $P = NP$?
- $P \neq EXP$, but is $NP = EXP$?
- $P \subseteq NP \cap$ co-NP, but is $P = NP \cap$ co-NP?
- Is $L = NL$?

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The Determinant Problem: Given an $n \times n$ matrix $M = (m_{ij})$, compute

$$\det(M) = \sum_{\pi \in S_n} (-1)^{\pi} \prod_{i=1}^{n} m_{i,\pi(i)},$$

where $(-1)^{\pi} = \begin{cases} -1 & \text{if } \pi \text{has odd parity,} \\ +1 & \text{if } \pi \text{ has even parity.} \end{cases}$

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Can be solved in $O(n^3)$, using the LU decomposition.

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The Permanent Problem: Given an $n \times n$ matrix $M = (m_{ij})$, compute

$$\mathrm{perm}(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} m_{i,\pi(i)}.$$

$$\mathrm{Perm} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh + ceg + bdi + afh$$

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Main difference $\det(M_1 \times M_2) = \det(M_1) \times \det(M_2)$ but $\mathrm{perm}(M_1 \times M_2) \neq \mathrm{perm}(M_1) \times \mathrm{perm}(M_2)$

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- How to differentiate strict exponential time from polynomial time?
- We cannot do that in most of the cases, we relate such answers to some of the open complexity questions.

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  Classes: #P

Function Problems: Examples

- Reachability: Given a direct graph G and two vertices u and v, find a path a path from u to v, if one exists.
- Eulerian cycle: Given a graph G, find an Eulerian cycle that traverses all the edges exactly once, if such a cycle exists.
- Factoring: Given an integer n, find its prime factors.
- Subset Sum: Given a sequence of positive integers $S = \{a_1, \ldots, a_n\}$ and $t \in \mathbb{Z}$, obtain $I \subseteq \{1, \ldots, n\}$ s.t. $\sum_{i \in I} a_i = t$.

Optimization Problems: Examples

- Max Cut: Given a graph G, find a partition a partition V into V_1, V_2, such that it maximizes the number of crossing edges between V_1 and V_2.
- Min Cut: Given a digraph G, find a partition V into V_1, V_2, such that it minimizes the number of edges going from V_1 to V_2.
- Shortest path: Given a connected and edge weighted digraph G and $s, t \in V(G)$, find a path between s and t minimizing the sum of the weights in the path.

Counting problems

- How many perfect matchings are there for a given bipartite graph?
- How many graph colorings using k colors are there for a particular graph G?
- What is the value of the permanent of a given matrix whose entries are 0 or 1?
- How many different variable assignments will satisfy a given general boolean formula?

More complexity classes?

Sure, plenty!
Look into the Complexity zoo.
547 classes the last time I've had a look!