Approximation and paremeterization: Complexity classes and hard problems

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Spring 2025

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  - There is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $\forall x \in \Sigma^* \kappa'(R(x)) \leq g(\kappa(x))$

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- We note $(L, \kappa) \leq^{fpt} (L', \kappa')$ when there is a FPT-reduction from (L, κ) to (L', κ')

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- Algorithm $\mathcal{A}'$ solves (L', κ') in $f'(\kappa'(y))p(|y|)$ time.
- Computing the FPT reduction R takes time $f_R(\kappa(x))p_R(|x|)$.
- Running $\mathcal{A}'$ on $y = R(x)$ solves (L, κ) in time

$$\begin{aligned} f_R(\kappa(x))p_R(|x|) + f'(\kappa'(R(x)))p(|R(x)|) \\ &\leq f_R(\kappa(x))p_R(|x|) + f'(g(\kappa(x))p(|R(x)|) \\ &\leq f_R(\kappa(x))p_R(|x|) + f'(g(\kappa(x))p(f_R(\kappa(x))p_R(|x|)) \\ &\leq f(\kappa(x))p(|x|) \end{aligned}$$

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- If $\mathcal{C}$ is a class of parameterized problems
  - (L, κ) is $\mathcal{C}$-hard if $\mathcal{C} \subseteq [(L, \kappa)]^{fpt}$.
  - (L, κ) is $\mathcal{C}$-complete if $(L, \kappa) \in \mathcal{C}$ and (L, κ) is $\mathcal{C}$-hard.

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  - (L, κ) is $\mathcal{C}$-complete if $(L, \kappa) \in \mathcal{C}$ and (L, κ) is $\mathcal{C}$-hard.
- $[(L, \kappa)]^{fpt}$ defines a class of parameterized problems for which (L, κ) is complete
- if (L, κ) is $\mathcal{C}$-complete and $\mathcal{C}$ is closed under FPT reductions, then $\mathcal{C} = [(L, \kappa)]^{fpt}$

FPT-equivalent problems

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  Exercise

The class paraNP

- Let (L, κ) be a parameterized problem
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- If $L \in$ NP, for each parameterization κ, $(L, \kappa) \in$ paraNP
  p-Clique, p-Vertex Cover, ... belong to paraNP.

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Theorem

If $(L, \kappa) \in$ paraNP is not trivial and has a NP-complete slice, then (L, κ) is paraNP-complete under FPT reductions.

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- paraNP-completeness separates *all slices* in P from *a slice* is NP-hard.

The class XP

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- p-CLIQUE, p-VERTEX COVER, p-HITTING SET, p-HITTING SET, p-DOMINATING SET belong to XP.
- XP is the counterpart of EXP in classic complexity.

XP-complete problems

p-Exp-DTM-Halt
Input: A deterministic TM $\mathbb{M}$, $x \in \Sigma^*$ and an integer k,
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Question: Does $\mathbb{M}$ on input x stop in no more than $|x|^k$ steps?

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Theorem

p-Exp-DTM-Halt *is XP-complete but does not belong to FPT.*

Relationships among classes

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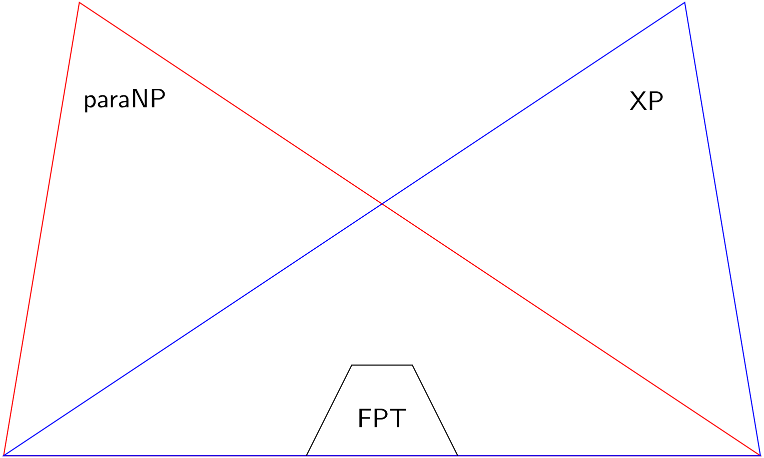
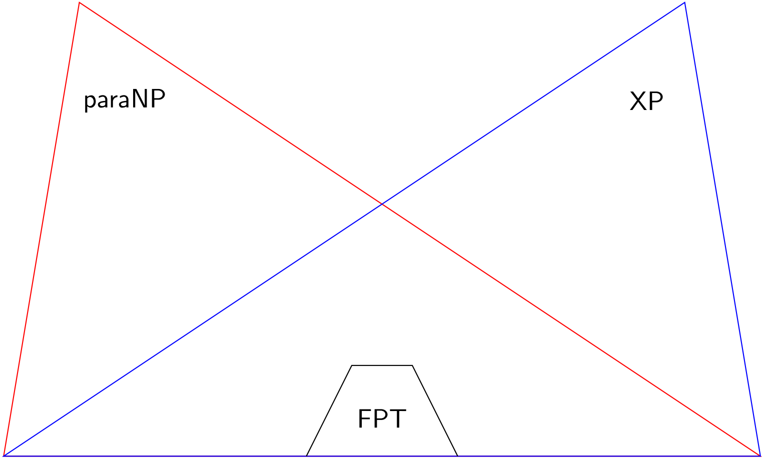


The W-hierarchy



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- Note that $\text{depth}(C) \geq \text{weft}(C)$

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p-WSAT(FAM)
Input: A circuit/formula C/F in family FAM and an integer k,
Parameter: k
Question: Is C/F k-satisfiable?

W-classes

Families of circuits/formulas

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$$\mathcal{C}_{t,d} = \{c \mid C \in \textsc{Circ} \text{ and } \text{weft}(C) \leq t \text{ and } \text{depth}(C) \leq d\}$$

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Theorem

- $W[P] \subseteq paraNP \cap XP$
- $W[SAT] \subseteq W[P]$
- *For* $i \geq 1$, $W[i] \subseteq W[SAT]$ *and* $W[i] \subseteq W[i+1]$

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Theorem

If $FPT = W[P]$ *then* CIRCUITSAT *for circuits with* n *inputs and* m *gates can be decided in* $2^{o(n)} m^{O(1)}$ *time.*

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  In fact the problem is $W[1]$-complete
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  In fact both problems are $W[2]$-complete

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- We wish to get results like:
  If there is an $f(k)\ n^{o(k)}$ time algorithm for problem XXX, then ETH fails.

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- Can we do it much faster, for example in time $O^*(c^{\sqrt{k}})$ or $O^*(c^{k/logk})$?

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Lemma

If VERTEX COVER can be solved in time $2^{o}(k)\ n^{O(1)}$, then ETH fails.

Proof.

There is a polynomial-time reduction from m-clause 3SAT to $O(m)$-vertex VERTEX COVER. The assumed algorithm would solve the latter problem in time $2^{o(m)}\ n^{O(1)}$, violating ETH. □

Efficient approximation schemes

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  Input: Instance $x, \epsilon > 0$
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- Efficient PTAS (EPTAS) running time is $f(1/\epsilon)|x|^{O(1)}$
- For some problems, there is a PTAS, but no EPTAS is known.

  Can we show that no EPTAS is possible?

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Lemma

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Proof.

Suppose an $f(1/\epsilon)\ n^{O(1)}$ time EPTAS exists.
Running this EPTAS with $\epsilon = 1/(k+1)$ decides if the optimum is at most/at least k. □

Parameterized complexity

- Possibility to give evidence that certain problems are not FPT.
- Parameterized reduction.
- The W-hierarchy.
- ETH gives much stronger and tighter lower bounds.
- PTAS vs. EPTAS
- Kernel lower bounds