Parameterized algorithms: Tree width and dynamic programming

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- A similar parameter measures closeness of a graph to a path: pathwidth.

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- For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v. $N[v] = N(v) + v$. $d(v) = |N(v)|$.
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- A graph is outerplanar if it can be drawn as a cycle with non-crossing chords.

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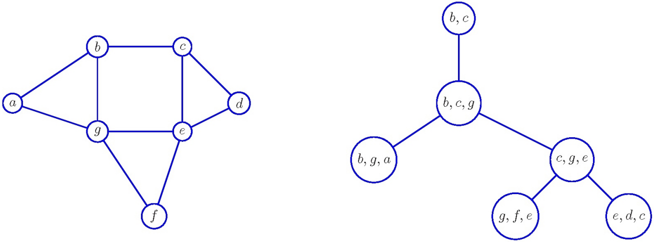

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- Equivalently the second condition can be expressed as:
  - For every $u, v \in V(T)$ and every vertex w on the path between u and v, $X_u \cap X_v \subseteq X_w$, and every vertex of G appears in at least one X_v .

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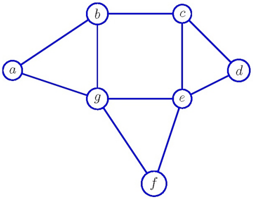

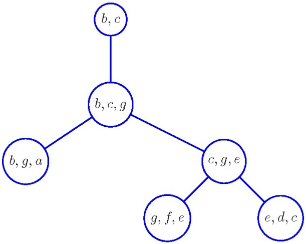

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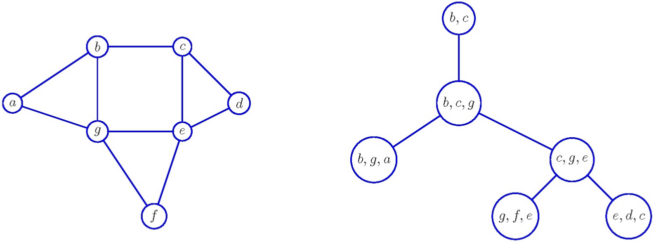

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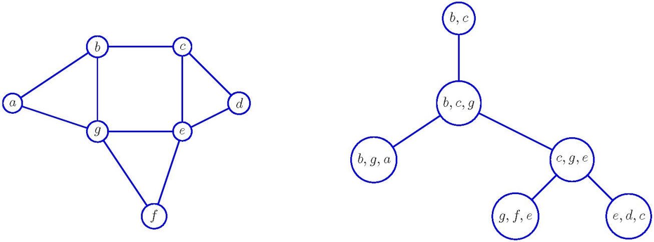

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- The sets X_v are called the bags of the tree decomposition.

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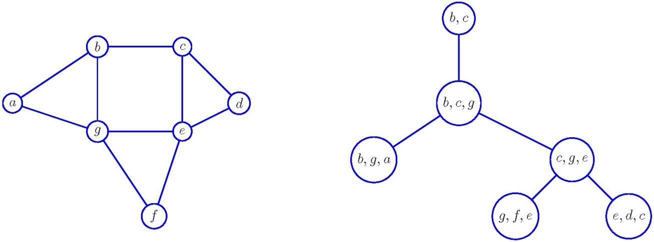
- The tree width ($tw(G)$) of a graph G is the minimum width over all tree decompositions of G.

A graph with tree width 2



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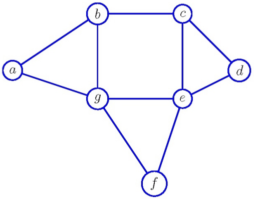

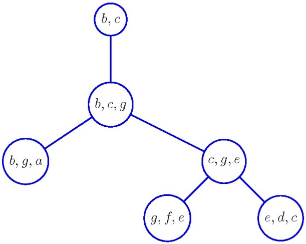

This graph is an outerplanar graph.

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- For K_n, the complete graph on n vertices, $tw(K_n) = n - 1$.

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- There is a polynomial time approximation algorithm that given a (G, k) decides in polynomial time whether $tw(G) \leq k$ and if so produces a tree decomposition of width $\leq r \cdot k$ and size $O(|V|)$, for some constant $r > 1$.

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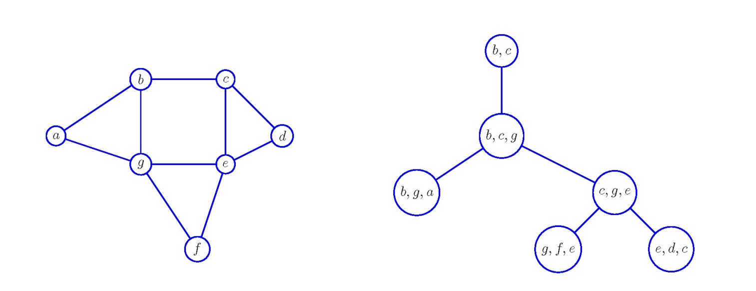
- Notice that X_v is a separator in G.



Exercise For this rooted tree decomposition. Draw the graphs associated to each node in the tree. What are the differences among parent-child graphs?

1. Tree width
2. Parameterizing by tree width
3. Nice tree decomposition
4. Min Vertex cover parameterized by treewidth

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    with $X_v \subseteq X_u$ and $|Xu| = |Xv| + 1$ (introduce)

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  - u has two children v and w with $X_u = X_v = X_w$ (join)

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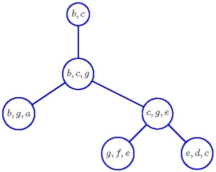

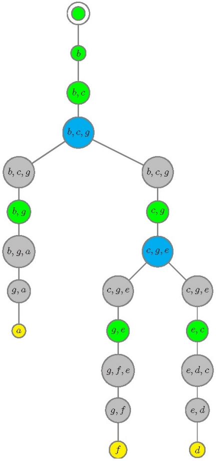

Nice tree decomposition

Lemma

Computing a rooted nice tree decomposition with width at most k, given a small tree decomposition of width at most k takes $O(kn)$ time.

Nice tree decomposition

- Nodes in the tree
  node u holds a subset of vertices X_u, and has a subgraph G_u associated to it.
- the root r has $X_r = \emptyset$ and $G_r = G$.
- nodes can be of four types:
  Start Introduce Forget Join

A parameterization for Vertex Cover?

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PROBLEM: $tw(G)$ cannot be computed in polynomial time (unless P=NP)!

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tw-Vertex Cover

Input: A graph G, a tree decomposition (T, X) of G and an integer k,

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tw-k-Vertex Coloring
Input: A graph G and integers w and k,
Parameter: $w + k$
Question: Does G have a vertex cover with k vertices and $tw(G) \leq w$?
The same kind of tw parameterization applies to other graph problems.

For some graph properties, an upper bound in terms of treewidth can be found. For such problems the parameter might be only w.

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- The value of $s_r(\emptyset)$ is the size of a minimum vertex cover of G.
- We deal with each type of node separately

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Claim

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The associated graph has just a node, therefore the values are correct.

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  All neighbors of x in $G(u)$ are in X_v and therefore in C since C is a vertex cover of $G[X_u]$.

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Proof.

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Let u be a *forget* node, let v be its unique child and assume that $\{v\} = X_v - X_u$.
Then for all $C \subseteq X_u$, $s_u(C) = \min\{s_v(C), s_v(C + x)\}$.

Proof.

- ($\geq$) Let C' be a min$\textsc{VC}$ of $G(u) = G(v)$ with $C' \cap X_u = C$.

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- $(\geq)$ Let C' be a minVC of $G(u) = G(v)$ with $C' \cap X_u = C$.
  - If $x \notin C'$ then $C' \cap X_u = C$ so $|C'| \geq s_v(C)$.

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  - If $x \notin C'$ then $C' \cap X_u = C$ so $|C'| \geq s_v(C)$.
  - If $x \in C'$ then similarly $|C'| \geq s_v(C + x)$.
- $(\leq)$ Let C_1 and C_2 be the $\textsc{vc}$s that determine $s_v(C)$ and $s_v(C + x)$ respectively. C_1, C_2 are $\textsc{vc}$ of $G(u)$ compatible with C, so $s_v(C) \leq \min\{|C_1|, |C_2|\}$.

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Join node

Join node

Claim

Let u be a join node of T with children v and w.
Then for all $C \subseteq X_u$: $s_u(C) = s_v(C) + s_w(C) - |C|$.

Proof.

Join node

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Let u be a join node of T with children v and w.
Then for all $C \subseteq X_u$: $s_u(C) = s_v(C) + s_w(C) - |C|$.

Proof.

- $(\geq)$ If C' is a vertex cover of $G(u)$ with $C' \cap X_u = C$, then $C' \cap V(v)$ is a vertex cover of $G(v)$ and $C' \cap V(w)$ is a vertex cover of $G(w)$. These sets share $|C|$ vertices.

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- ($\leq$) Two C-compatible vertex covers of $G(v)$ and $G(w)$ of size $s_v(C)$ and $s_w(C)$ can be combined to a vertex cover of $G(u)$ of size $s_v(C) + s_w(C) - |C|$.

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Min Vertex cover parameterized by treewidth

Theorem

Let (T, X) be a rooted nice tree decomposition of width w of a graph G on n vertices. In time $2^{w+1}n^{O(1)}$ the size of a minimum vertex cover of G can be computed.

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- We can construct a minimum vertex cover as well, by tracing back through the tree decomposition.

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Proof.

- We can construct a minimum vertex cover as well, by tracing back through the tree decomposition.
- $+ O(f(k)n)$ to get the tree decomposition if needed.

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