

Parameterized algorithms: Tree width and dynamic programming

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- 1 Tree width
- 2 Parameterizing by tree width
- 3 Nice tree decomposition
- 4 Min Vertex cover parameterized by treewidth

Graph parameters

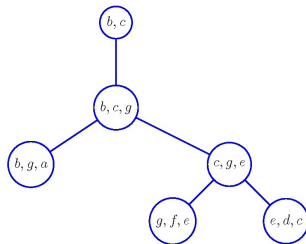
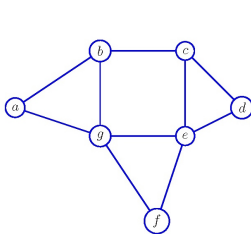
- For a given graph G we can consider graph measures as candidates for parameters as diameter, degree, min vertex cover, \dots , or a combination of them.
- Many hard graph problems can be solved in polynomial time in trees.
- We are going to explore a parameter that measures the closeness of a graph to a tree: **treewidth**.
- A similar parameter measures closeness of a graph to a path: **pathwidth**.

Recall some graph notation

- For a graph G and $v \in V(G)$, $G - v$ denotes the graph obtained by deleting v (and all incident edges).
- For a set S , $S + v$ denotes $S \cup \{v\}$, and $S - v$ denotes $S \setminus \{v\}$.
- For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v . $N[v] = N(v) + v$. $d(v) = |N(v)|$.
- For a graph $G = (V, E)$, $\delta(G) = \min_{v \in V} d(v)$, and $\Delta(G) = \max_{v \in V} d(v)$.
- A **tree** is a connected graph without cycles.
- A **forest** is a graph without cycles.
- A **unicyclic** graph has only one cycle.
- A graph is **outerplanar** if it can be drawn as a cycle with non-crossing chords.

Tree decomposition

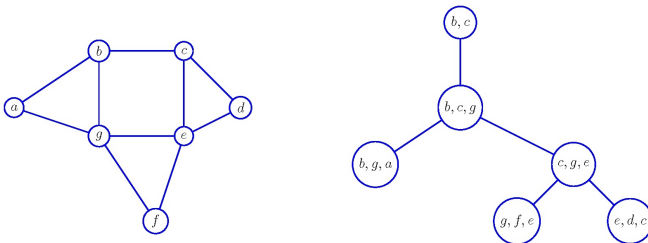
- A **tree decomposition** of a graph G is a tuple (T, X) where T is a tree and $X = \{X_v \mid v \in V(T)\}$ is a set of subsets of $V(G)$ such that:
 - For every $xy \in E(G)$, there is a $v \in V(T)$ with $\{x, y\} \subseteq X_v$.
 - For every $x \in V(G)$, the subgraph of T induced by $X^{-1}(x) = \{v \in V(T) \mid x \in X_v\}$ is non-empty and connected.



Tree decomposition

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- Equivalently the second condition can be expressed as:
 - For every $u, v \in V(T)$ and every vertex w on the path between u and v , $X_u \cap X_v \subseteq X_w$, and every vertex of G appears in at least one X_v .

Tree decomposition

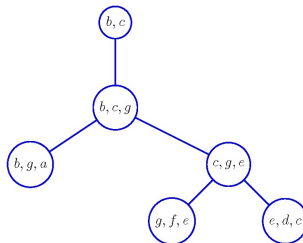
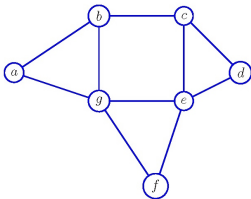


- To distinguish between vertices of G and T , the vertices of T are called **nodes**.
- The sets X_v are called the **bags** of the tree decomposition.

Tree width

- The **width** of a tree decomposition (T, X) for G is $\max_{v \in V(T)} |X_v| - 1$.
- The **tree width** ($tw(G)$) of a graph G is the minimum width over all tree decompositions of G .

A graph with tree width 2



This graph is an **outerplanar** graph.

Tree width of some graphs

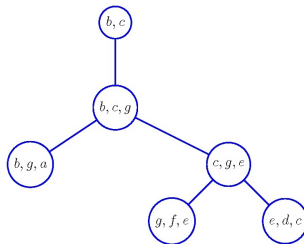
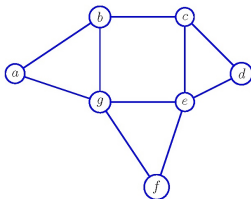
- $tw(G) = 0$ iff $E(G) = \emptyset$.
- If G is a forest $tw(G) \leq 1$.
 - Consider the tree obtained from G by subdividing every edge $uv \in E(G)$ with a new vertex w_{uv} .
 - Set $X_u = \{u\}$ for all $u \in V(G)$, and $X_{w_{uv}} = \{u, v\}$ for every $uv \in E(G)$.
- If G is outerplanar then $tw(G) \leq 2$.
 - Let G' be a graph obtained after triangulating arbitrarily the face of G with more than 3 sides preserving outerplanarity .
 - T is the dual of G' : a node per face and connecting two nodes if their faces share an edge.
 - Associate to every node the three vertices in the corresponding face.
- For K_n , the complete graph on n vertices, $tw(K_n) = n - 1$.

Tree width complexity

- Deciding if a graph has treewidth k is NP-complete.
- Given a graph $G = (V, E)$ and $k \leq |V| - 1$, computing a tree decomposition with width at most k (if it exists) takes $O(f(k)n)$ time. Furthermore the computed tree has $O(|V|)$ vertices.
- There is a polynomial time approximation algorithm that given a (G, k) decides in polynomial time whether $tw(G) \leq k$ and if so produces a tree decomposition of width $\leq r \cdot k$ and size $O(|V|)$, for some constant $r > 1$.

Notation

- Let (T, X) be a tree decomposition of G of width k .
 Such a tree decomposition can be computed (if possible) by an FPT/constant approximation algorithm
- Make T into a rooted tree by choosing a root $r \in V(T)$, and replacing edges by arcs in such a way that every node points to its parent.
- For $v \in V(T)$, $R_T(v)$ denotes the nodes in the subtree rooted at v (including v).
- For $v \in V(T)$, $V(v) = X(R_T(v))$, and $G(v) = G[V(v)]$.
 Thus, we have an induced subgraph associated to each node.
- Notice that X_v is a separator in G .



Exercise For this rooted tree decomposition. Draw the graphs associated to each node in the tree. What are the differences among parent-child graphs?

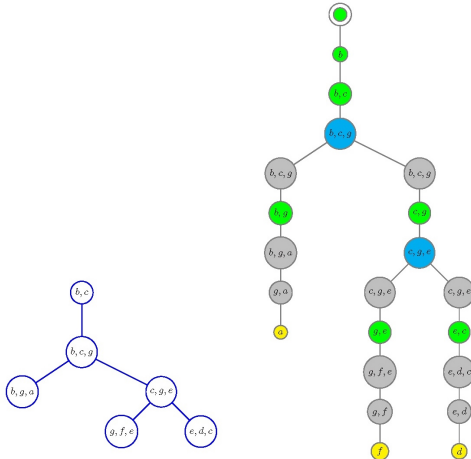
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Nice tree decomposition

- A nice tree decomposition is a variant in which the structure of the nodes is simpler.
- A rooted tree decomposition (T, X) is **nice** if for every $u \in V(T)$
 - $|X_u| = 1$ (start)
 - u has one child v
 - with $X_u \subseteq X_v$ and $|X_u| = |X_v| - 1$ (forget)
 - with $X_v \subseteq X_u$ and $|X_u| = |X_v| + 1$ (introduce)
 - u has two children v and w with $X_u = X_v = X_w$ (join)

Nice tree decomposition



Nice tree decomposition

Lemma

Computing a rooted nice tree decomposition with width at most k , given a small tree decomposition of width at most k takes $O(kn)$ time.

Nice tree decomposition

- Nodes in the tree
node u holds a subset of vertices X_u , and has a subgraph G_u associated to it.
- the root r has $X_r = \emptyset$ and $G_r = G$.
- nodes can be of four types:

Start

Introduce

Forget

Join

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A parameterization for Vertex Cover?

TW-K-VERTEX COVER

Input: A graph G and an integer k ,

Parameter: $tw(G) + k$

Question: Does G have a vertex cover with k vertices?

PROBLEM: $tw(G)$ cannot be computed in polynomial time
(unless $P=NP$)!

Equivalent parameterizations for Vertex Coloring

TW-VERTEX COVER

Input: A graph G , a tree decomposition (T, X) of G
and an integer k ,

Parameter: $\text{width}(T, X) + k$

Question: Does G have a vertex cover with k vertices?

TW-K-VERTEX COLORING

Input: A graph G and integers w and k ,

Parameter: $w + k$

Question: Does G have a vertex cover with k vertices and $\text{tw}(G) \leq w$?

The same kind of tw parameterization applies to other graph problems.

For some graph properties, an upper bound in terms of treewidth can be found. For such problems the parameter might be only w .

The algorithm for tw-Vertex Cover

- Obtain a rooted nice tree decomposition (T, X) of G
- For each node $v \in V(T)$,
we keep a value $s_v(C)$, for each $C \subseteq X_v$, holding the
minimum size of a vertex cover C' of $G(v)$ with $C' \cap X_v = C$
if such a C' exists, and $s_v(C) = \infty$ otherwise.
- The value of $s_r(\emptyset)$ is the size of a minimum vertex cover of G .
- We deal with each type of node separately

Start node

Claim

Let u be a leaf of T with $X_u = \{x\}$, $s_u(\{x\}) = 1$ and $s_u(\emptyset) = 0$.

The associated graph has just a node, therefore the values are correct.

Introduce node

Claim

Let u be an *introduce* node, let v be its unique child and assume that $\{x\} = X_u - X_v$.

Then for all $C \subseteq X_u$

- If C is not a vertex cover of $G[X_u]$ then $s_u(C) = \infty$.
- If C is a vertex cover of $G[X_u]$ and $x \in C$ then $s_u(C) = s_v(C - x) + 1$.
- If C is a vertex cover of $G[X_u]$ and $x \notin C$ then $s_u(C) = s_v(C)$.

All neighbors of x in $G(u)$ are in X_v and therefore in C since C is a vertex cover of $G[X_u]$.

Forget node

Claim

Let u be a *forget* node, let v be its unique child and assume that $\{v\} = X_v - X_u$.

Then for all $C \subseteq X_u$, $s_u(C) = \min\{s_v(C), s_v(C + x)\}$.

Proof.

- (\geq) Let C' be a minVC of $G(u) = G(v)$ with $C' \cap X_u = C$.
 - If $x \notin C'$ then $C' \cap X_u = C$ so $|C'| \geq s_v(C)$.
 - If $x \in C'$ then similarly $|C'| \geq s_v(C + x)$.
- (\leq) Let C_1 and C_2 be the VCs that determine $s_v(C)$ and $s_v(C + x)$ respectively. C_1, C_2 are VC of $G(u)$ compatible with C , so $s_v(C) \leq \min\{|C_1|, |C_2|\}$.



Join node

Claim

Let u be a join node of T with children v and w .

Then for all $C \subseteq X_u$: $s_u(C) = s_v(C) + s_w(C) - |C|$.

Proof.

- (\geq) If C' is a vertex cover of $G(u)$ with $C' \cap X_u = C$, then $C' \cap V(v)$ is a vertex cover of $G(v)$ and $C' \cap V(w)$ is a vertex cover of $G(w)$. These sets share $|C|$ vertices.
- (\leq) Two C -compatible vertex covers of $G(v)$ and $G(w)$ of size $s_v(C)$ and $s_w(C)$ can be combined to a vertex cover of $G(u)$ of size $s_v(C) + s_w(C) - |C|$.



Min Vertex cover parameterized by treewidth

Theorem

Let (T, X) be a rooted nice tree decomposition of width w of a graph G on n vertices. In time $2^{w+1}n^{O(1)}$ the size of a minimum vertex cover of G can be computed.

Proof.

- We can construct a minimum vertex cover as well, by tracing back through the tree decomposition.
- + $O(f(k)n)$ to get the tree decomposition if needed.

