Parameterization: basics classes and algorithms

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1. Parameterization
2. Bounded search tree
3. Kernelization

Three NP complete problems

VERTEX COLORING

Given a graph G and an integer k,
$\exists\sigma : V(G) \to \{1,\dots,k\} \mid \forall\{u,v\} \in E(G)\sigma(u) \neq \sigma(v)$?

INDEPENDENT SET

Given a graph G and an integer k,
$\exists S \subseteq V(G) \mid |S| = k$ and $\forall\{u,v\} \in E(G)|\{u,v\} \cap S| \leq 1$?

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Let's look to exact algorithms.

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A better algorithm?

Vertex Cover

function $\textsc{algVC}(G, k)$
    if $|E(G)| = 0$ **then**
        return true
    end if
    if $k = 0$ **then**
        return false
    end if
    Select and edge $e = \{u, v\} \in E(G)$
    return $\textsc{algVC}(G - u, k - 1)$ or $\textsc{algVC}(G - v, k - 1)$
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Correctly solves the problem and takes time $O(m2^k)$

Algorithms cost

Given a graph G and an integer k:

- VERTEX COLORING: $O(n^2 k^n)$
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Objective: Find slices of the problem having efficient algorithms

Slice: The instances with a particular value of a parameter

Natural small parameters

- VLSI design: the number of layers in a chip is below 10.
- Biology: DNA chains in many cases have path width below 11
- Robotics: The robot movements have small dimension
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- Compilers: Type compatibility is usually EXP-complete, however typical type declaration have small depth
- Optimization problem: the measure of the optimal solution is small
- A problem might have more than one parameter of interest and the behavior with respect to different parameters might be different.

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- Parameterized problems are decision problems together with a parameterization.
- A problem can be analyzed under different parameterizations.

Parameterized problem: An example

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Given a CNF formula F,
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- Consider $\kappa : \Sigma^* \to \mathbb{N}$

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  P#VAR-SAT
  Input: A CNF formula F,
  Parameter: The number of variables in F
  Question: is there a satisfying assignment for F?

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  PMAX#LIT-SAT
  Input: A CNF formula F
  Parameter: The maximum number of literals in a clause in F
  Question: is there a satisfying assignment for F?

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Graph problems and parameters

- Let G be a graph and k a natural number.
- The function $\kappa(G, k) = k$ is used to define the parameterized problems
  - p-Independent Set
  - p-Vertex Coloring
  - p-Vertex Cover
  - p-Dominating Set
  - p-Clique
  - etc.

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- For problems on graphs we can use other graph properties to define graph parameters like max degree or diameter.
  Or any other graph parameter of interest.

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- We have show that there is an algorithm for VERTEX COVER requiring $O(|E(G)|2^k)$ time
  P-VERTEX COVER belongs to FPT!

Other classes (hard parameterized problems)

- paraNP
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  - p-CLIQUE, p-VERTEX COVER, p-HITTING SET, p-DOMINATING SET belong to XP.

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- In between FPT and those classes it is placed the W-hierarchy W[1], W[2] ... defined through logic/circuit characterizations

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- Such type of recursive algorithm is called a bounded search tree algorithm.
- If we have a constant bound on the number of recursive calls, depth bounded by the parameter, and polynomial cost per call, the resulting algorithm is an FPT algorithm.

Hitting Set

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Input: a collection of subsets $\mathcal{S} = (S_1, \ldots, S_m)$ of $U = \{1, \ldots, n\}$ and an integer k.

Question: $\exists A \subseteq U \mid |A| = k$ and $\forall X \in \mathcal{S}\ |X \cap A| \geq 1$?

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P-HITTING SET

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p-Hitting Set

function $\text{ALGHS}(U, \mathcal{S}, k)$
  if $|\mathcal{S}| = 0$ **then**
    return true
  end if
  if $k = 0$ **then**
    return false
  end if
  Select a set $X \in \mathcal{S}$
  for all $v \in X$ **do**
    $V = U - \{v\}$; $\mathcal{S}_v = \{X \in \mathcal{S} \mid v \notin X\}$
    if $\text{ALGHS}(V, \mathcal{S}_v, k-1)$ **then**
      return (true)
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- using the above inequalities it is easy to prove that $T(s, k, d) \leq (2d^k - 1)c\,s$.

p-Hitting Set

Lemma

p-Hitting Set *belongs to FPT*

Bounded search tree technique

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- When the parameter is unbounded the algorithms take exponential time.
- We get FPT algorithm because the depth and/or branching of the recursion are function of the parameter.
- This algorithmic technique is called bounded search trees.
- As a design tool we have to look for parameterizations allowing a recursive algorithm with those characteristics.

A faster algorithm for p-VC

Recall some notation

A faster algorithm for p-VC

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- For a graph G and $v \in V(G)$, $G - v$ denotes the graph obtained by deleting v (and all incident edges).
- For a set S, $S + v$ denotes $S \cup \{v\}$, and $S - v$ denotes $S \setminus \{v\}$.
- For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v. $N[v] = N(v) + v$. $d(v) = |N(v)|$.
- For a graph $G = (V, E)$, $\delta(G) = \min_{v \in V} d(v)$, and $\Delta(G) = \max_{v \in V} d(v)$.

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- The recursion can skip a branching!

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- If $\delta(G) \geq 2$, $x \geq 2$. The recursion can jump to a smaller problem in one step!

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A faster algorithm for p-VC

- FASTVC:

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- FastVC:
  - If there is a vertex with degree one, use recursion of degree 1 vertices.
  - If there is a vertex with degree two, use recursion of degree 2 vertices.
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- How to get a bound in the cost? Guess and prove by induction!

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Theorem

The search tree corresponding to FASTVC *has at most* 1.47^k *leaves.*

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- If $k \geq 1$, then there are 3 cases:

A faster algorithm for p-VC

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- G contains a degree $d \geq 3$ vertex, branch into two cases $(G', k - 1)$ and $(G'', k - d)$. By induction, the total number of leaves is at most $1.47^{k-1} + 1.47^{k-3} \leq 1.47^k$.

☺

A faster algorithm for p-VC

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Theorem

FastVC *has cost* $O(1.47^k p(n+m))$, *for some polynomial* p *besides the constant in* O *is also constant with respect to the parameter* k.

1. Parameterization

2. Bounded search tree

3. Kernelization

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- An algorithm that computes x' and solves by brute force this instance has cost
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  So, it is an FPT algorithm provided the problem is decidible.

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- How big is (G', k')?

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If G has a vertex cover with $\leq k$ vertices and all the vertices have degree $\leq k$, $|E(G')| \leq k^2$.

- So, we can filter as NO instances those leading to reduced instances with a high number of edges!

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Obs 3

If G has a vertex cover with $\leq k$ vertices and all the vertices have degree $\leq k$, $|E(G')| \leq k^2$.

- So, we can filter as NO instances those leading to reduced instances with a high number of edges!
- By Obs 3, if $|E(G')| > k^2$, we replace (G', k') by a trivial small NO-instance, which is again equivalent.

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Theorem

Let (G, k) be an instance to p-VC. *In polynomial time we can obtain an equivalent* p-VC *instance (G', k') with $|V(G')|, |E(G')| \leq O(k^2)$.*

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Kernelization algorithm

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    $z = \textsc{Ker-P}(x)$
    return $(\textsc{Alg-P}(z))$
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Kernelization algorithm

- Assume that $\textsc{Ker-P}$ is a polynomial time algorithm computing a kernel for a given instance of problem P and that $\textsc{Alg-P}$ is an exact (exponential time) algorithm for P.

  function $\textsc{AlgKernel-P}(x)$
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- $\textsc{AlgKer-p-VC}$ is an FPT algorithm for P.

A kernelization algorithm for p-VC

function $\textsc{AlgKernel-p-VC}(G, k)$

$(G', k') =$ Iteratively remove isolated vertices and vertices with degree at least $k+1$, decreasing the parameter by one in the second case.

if $|E(G')| > k^2$ **then return** $\textsc{no}$

end if

for each $S \subseteq V'$ with $|S| = k'$ **do**

  if S is a vertex cover **then return** $\textsc{si}$

  end if

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$\textsc{AlgKernel-p-VC}$ runs in $O(n^c + k^{2k}k^2) = O(n^c) + O(k^{2k+2})$

p-MaxSat

P-MAXSAT

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Recall that the size of a CNF formula is the sum of clause lengths (# literals); we ignore as usual log-factors.

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Obs 2

Let F_s be obtained from formula F by removing all $\ell < k$ long clauses. $(F_s, k - \ell)$ and (F, k) are equivalent.

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If F contains at least $2k$ clauses, (F, k) is a YES instance of p-MaxSat.

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Proof.

Take an arbitrary truth assignment x and its complement $\overline{x}$ obtained by flipping all variables. Every clause of F is satisfied by x or by *overlinex* (or by both). The one that satisfies most clauses satisfies at least k clauses. ☺

A kernelization algorithm for p-MaxSat

```
function AlgKernel-p-MaxSat(F, k)
    Remove from F all t trivial clauses and set k = k − t
    if F has at least k long clauses then return YES
    end if
    Remove from F all ℓ long clauses and set k = k − ℓ
    if F has at least 2k clauses then return YES
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    for each set of k clauses do
        for each selection of one literal per clause in the set do
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- *Crown decomposition* is a general kernelization technique based on some results on matchings.

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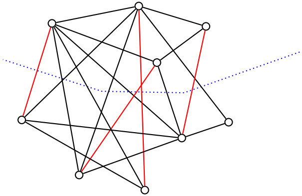
- For disjoint vertex subsets U, W of a graph G, M is a matching of U into W if every edge of M connects a vertex of U and a vertex of W and every vertex of U is an endpoint of some edge of M.
  We also say that M saturates U.



If M saturates U, $|U| \leq |W|$

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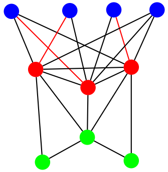

Computing a crown decomposition

Theorem (König's theorem)

In every undirected bipartite graph the size of a maximum matching is equal to the size of a minimum vertex cover.

Theorem (Hall's theorem)

Let $G = (V_1, V_2, E)$ be an undirected bipartite graph. G has a matching saturating V_1 iff for all $X \subseteq V_1$, we have $|N(X)| \geq |X|$.

Can you obtain a minimum vertex cover in a bipartite graph in polynomial time?

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Computing a crown decomposition

Theorem ((Hopcroft-Karp, SIAM J. Computing 2, 225–231 (1973))

Let $G = (V_1, V_2, E)$ be an undirected bipartite graph on n vertices and m edges. Then we can find a maximum matching as well as a minimum vertex cover of G in time $O(m\sqrt{n})$. Furthermore, in time $O(m\sqrt{n})$ either we can find a matching saturating V_1 or an inclusion-wise minimal set $X \subseteq V_1$ such that $|N(X)| < |X|$.

Crown lemma

Lemma

Let $G = (V, E)$ be a graph without isolated vertices and with at least $3k + 1$ vertices. There is a polynomial-time algorithm that either

- *finds a matching of size $k + 1$ in G; or*
- *finds a crown decomposition of G.*

Proof

We compute a maximal matching M in G.
If $|M| \geq k + 1$, we are done.

Now, $1 \leq |M| \leq k+1$.

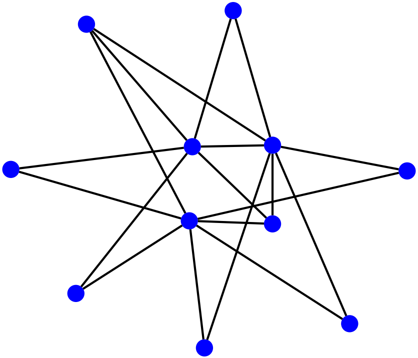
Let V_M be the end points of M and $I = V - V_M$.

- M is a maximal matching, so I is an independent set.
- Let G_{I,V_M} be the bipartite subgraph induced in G by I and V_M.
- In polynomial time, we compute a minimum size vertex cover X and a maximum matching M' in G_{I,V_M}.
- If $|M'| \geq k$, we are done. From now on, $|M'| \leq k$ and also $|X| \leq k$.
- If $X \cap V_M = \emptyset$, $X = I$. Then, $|I| = |X| \leq k$ and $|V| = |I| + |X| \leq k + 2k \leq 3k$!
- Then, $X \cap V_M \neq \emptyset$

- We obtain a crown decomposition (C, H, R) as follows.
- Since $|X| = |M'|$, every edge of the matching M' has exactly one endpoint in X.
- Let M^* be the subset of M' such that every edge from M^* has exactly one endpoint in $X \cap V_M$ and let V_{M^*} denote the set of endpoints of edges in M^*.
- Set head $H = X \cap V_M = X \cap V_{M^*}$, crown $C = V_{M^*} \cap I$, and the remaining part is R.
- C is an independent set and, by construction, M^* is a matching of H into C.
- Since X is a vertex cover of G_{I,V_M}, every vertex of C can be adjacent only to vertices of H.

End proof

An example with $k = 3$



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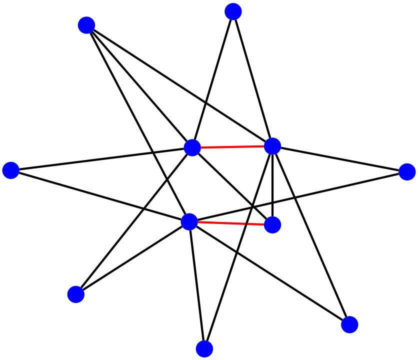

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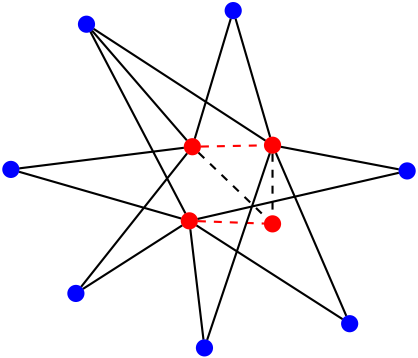

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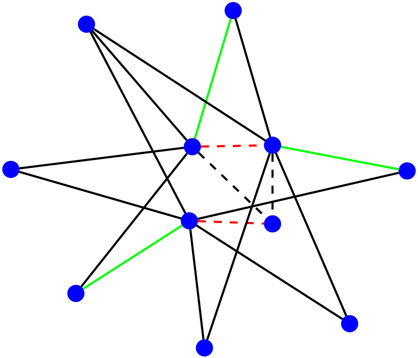

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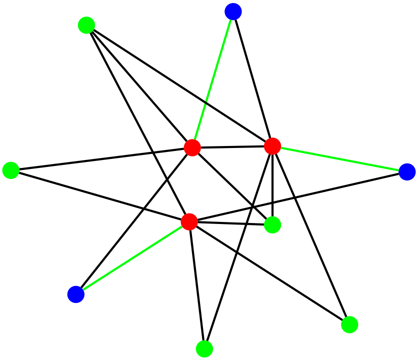

Crown decomposition: Vertex cover

Consider a Vertex Cover instance (G, k).

- By an exhaustive application of the isolated vertex reduction rule, we may assume that G has no isolated vertices.
- If $|V(G)| > 3k$, we use the crown lemma to get either
- a matching of size $k + 1$, (so (G, k) is a no-instance) or a crown decomposition C, H, R.

Crown decomposition: Vertex cover

From the crown decomposition C, H, R of G, let M be a matching of H into C.

- The matching M witnesses that, for every vertex cover X of G, X contains at least $|M| = |H|$ vertices of $H \cap C$ to cover the edges of M.
- H covers all edges of G that are incident to $H \cup C$.
- So, there exists a minimum vertex cover of G that contains H, and we may reduce (G, k) to $(G - H, k - |H|)$.
- Further, in $(G - H, k - |H|)$, $c \in C$ is isolated and can be eliminated.

Crown decomposition: Vertex cover

As the crown lemma promises that $H \neq \emptyset$, we can always reduce the graph as long as $|V(G)| > 3k$.

Lemma

Vertex Cover admits a kernel with at most $3k$ vertices.

Crown decomposition: Max SAT

Lemma

Max SAT admits a kernel with at most k variables and $2k$ clauses.

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  - return NO if $|X| > f(k)$, for minimization problems, or
  - return YES if $|X| > f(k)$, for maximization problems.

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- Hardness notion?
- We would like to get a kernel as small as possible.
- Statements like: (L, κ) does not admit a linear (quadratic) kernel unless some complexity assumption fails are the kind of results showing kernelization hardness.