# Linear Programming approximation: Primal Dual algorithms

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Spring 2024

1. Primal-Dual algorithms

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- If at some point objective functions match, we have found an optimal solution.
- If at some point relaxed complementary slackness holds, for some $r$, we have found a $r$-approximate solution.

# Bipartite graphs

MAXIMUM WEIGHT MATCHING IN BIPARTITE GRAPHS) (MWM-BG)

Given a bipartite graph $G = (A, B, E)$ and a weight function $w : E \to R$ find a matching of maximum weight where the weight of matching $M$ is given by $w(M) = \sum_{e \in M} w(e)$.

MINIMUM WEIGHT PERFECT MATCHING ON BIPARTITE GRAPHS)(mWPM-BG)

Given a bipartite graph $G = (A, B, E)$ and a weight function $w : E \to R \cup \infty$ find a matching of maximum weight where the weight of matching $M$ is given by $w(M) = \sum_{e \in M} w(e)$.

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Is MWM-BG $\leq$ mWPM-BG? YES!, even for complete bipartite graphs with $|A| = |B|$!

# ILP for MWPM-BG

$$\begin{aligned}
\min \quad & \sum_{a\in A, b\in B} w(a,b) x_{a,b} \\
\text{s.t.} \quad & \sum_{b\in B} x_{a,b} = 1 \quad \forall\, a \in A \\
& \sum_{a\in A} x_{a,b} = 1 \quad \forall\, b \in B \\
& x_{a,b} \in \{0,1\} \quad \forall\, a \in A, b \in B
\end{aligned}$$

In the LP relaxation, the last changes to $x_{a,b} \geq 0 \quad \forall\, a \in A, b \in B$

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Primal

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\begin{aligned}
\max \quad & \sum_{a\in A} y_a + \sum_{b\in B} y_b \\
\text{s.t.} \quad & y_a + y_b \leq w(a,b) \quad \forall\, a \in A, b \in B \\
& y_a \geq 0 \quad \forall\, a \in A \\
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$$w(M) = \sum_{(a,b)\in M} w(a, b) \geq \sum_{(a,b)\in M} (\hat{y}_a + \hat{y}_b) \geq \sum_{a\in A} \hat{y}_a + \sum_{b\in B} \hat{y}_b$$

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If all edges in $M$ are tight equality holds and $M$ is optimal.

# MWPM-BG: Primal dual algorithm

- The primal dual algorithm starts with a dual feasible solution, and a matching.
- At each time step it improves the number of tight edges and the weight of the matching, until the matching is perfect.
- At this point an optimal solution has been found.

# MWPM-BG: Primal dual algorithm

**function** PRIMAL-DUAL MWPM-BG$(A, B, E, w)$
  $y_b = 0$, for $b \in B$ and $y_a = \min_b w(a, b)$, for $a \in A$
  $E' =$ set of tight edges
  $M =$ max cardinality matching in $G' = (A, B, E')$
  **while** $M$ is not a perfect matching **do**
    $\overrightarrow{E} = \{e \in E' e \notin M \ (as \overrightarrow{AB})\} \cup \{e \in M \ (as \overrightarrow{BA})\}$
    $D = (A \cup B, \overrightarrow{E})$     % a directed graph.
    $L = \{v \in A \cup B \mid v$ is reachable in $D$ from an unmatched vertex in $A\}$
    $\epsilon = \min_{a \in A \cap L}(w(a, b) - y_a - y_b)$
    $y_a = y_a + \epsilon$, for $a \in A \cap L$ and $y_b = y_b - \epsilon$, for $b \in B \cap L$
    $E' =$ set of tight edges
    $M =$ max cardinality matching in $G' = (A, V, E')$
  **return** M

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**Claim**

After one iteration of the while loop

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**Theorem**

*Algorithm* PRIMAL-DUAL MWPM-BG *terminates in* $O(|A \cup B|^3)$ *iterations.*

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- We know how to relax the IP formulation as LP problem

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- We know how to formulate VC as an IP problem
- We know how to relax the IP formulation as LP problem
- We know how to compute the dual of the LP problem

# Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

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- $s^* \leq$ opt

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- So, relaxed complementary slackness conditions hold for $r = 2$. A 2-approximation ☺.

# Primal-Dual for weighted vertex cover

WVC

Given a vertex weighted graph $G = (V, A, c)$ we want to find a set $S \subset V$ with minimum weight, so that every edge in $G$ has at least one end point in $S$.

- The problem is NP-hard and belongs to NPO.
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- Restrictions: for every edge $(i, j) \in E$, $x_i + x_j \geq 1$
- $x_i \in \{0, 1\}$
- The IP can be computed in polytime.

# Weighted vertex cover: LP relaxation

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- Cost of the solution? $x$, $z$ are feasible. Relaxed complementary slackness conditions?

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- So, relaxed complementary conditions hold for $r = 2$ and we have a 2-approximation for WVC.

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- Trying to get the highest (the best) lower bound that we can get for the primal minimization objective.
  In general, this step can be implemented solving another LP program!
- We can also increase edge variables one by one. This leads to another primal-dual approximation algorithm PRICING METHOD

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- A vertex is tight with respect to a pricing $z$ if $\sum_{i \in e} z_e = c_i$.

# Pricing algorithm

Set prices and find vertex cover simultaneously.

**function** PRICING WVC($G, c$)
  $S = \emptyset$;
  **for** $e \in E$ **do**
    $z[e] = 0$ % initial price is 0
  **while** there is $(i, j) \in E$ so that neither $i$ nor $j$ is tight **do**
    select such an edge $e = (i, j)$
    Increase $z[e]$ until $i$ or $j$ became tight.
    Add to $S$ the vertex (vertices) that became tight.
  **return** S

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- Follows directly from primal-dual arguments.
- However, PRICING WVC is a greedy algorithm.
- No LP solver has been used!