# Linear Programming approximation: Primal Dual algorithms

Maria Serna

Spring 2024

AA-GEI: Approx, Param and Stream Linear Programming approximation: Primal Dual algorithms

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Primal-Dual algorithms Matchings in bipartite graphs Approximating Vertex Cover Weighted Vertex Cover Pricing method

## Primal dual algorithms

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- If at some point objective functions match, we have found an optimal solution.
- If at some point relaxed complementary slackness holds, for some *r*, we have found a *r*-approximate solution.

## Bipartite graphs

## MAXIMUM WEIGHT MATCHING IN BIPARTITE GRAPHS) (MWM-BG)

Given a bipartite graph G = (A, B, E) and a weight function  $w : E \to R$  find a matching of maximum weight where the weight of matching M is given by  $w(M) = \sum_{e \in M} w(e)$ .

## MINIMUM WEIGHT PERFECT MATCHING ON BIPARTITE GRAPHS)(mWPM-BG)

Given a bipartite graph G = (A, B, E) and a weight function  $w : E \to R \cup \infty$  find a matching of maximum weight where the weight of matching M is given by  $w(M) = \sum_{e \in M} w(e)$ .

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Is MWM-BG  $\leq$  mWPM-BG? YES!, even for complete bipartite graphs with |A| = |B|!

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## ILP for MWPM-BG

$$\begin{array}{ll} \min & \sum_{a \in A, b \in B} w(a, b) x_{a, b} \\ \text{s.t.} & \sum_{b \in B} x_{a, b} = 1 \quad \forall \, a \in A \\ & \sum_{a \in A} x_{a, b} = 1 \quad \forall \, b \in B \\ & x_{a, b} \in \{0, 1\} \quad \forall \, a \in A, b \in B \end{array}$$

In the LP relaxation, the last changes to  $x_{a,b} \ge 0 \quad \forall a \in A, b \in B$ 

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## ILP for MWPM-BG: The dual of the relaxed LP

#### Primal

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s.t. 
$$\sum_{b \in B} x_{a,b} = 1 \quad \forall a \in A$$
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The dual has a variable for each vertex  $y_a$ ,  $y_b$  and the form

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The dual has a variable for each vertex  $y_a$ ,  $y_b$  and the form

$$\begin{array}{ll} \max & \sum_{a \in A} y_a + \sum_{b \in B} y_b \\ \text{s.t.} & y_a + y_b \leq w(a,b) \quad \forall \, a \in A, \, b \in B \\ & y_a \geq 0 \quad \forall \, a \in A \\ & y_b \geq 0 \quad \forall \, b \in B \end{array}$$

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## ILP for MWPM-BG: tight edges

An edge e = (a, b) is tight, for a dual feasible solution y, if  $y_a + y_b = w(e)$ .

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$$w(M) = \sum_{(a,b)\in M} w(a,b) \ge \sum_{(a,b)\in M} (\hat{y}_a + \hat{y}_b) \ge \sum_{a\in A} \hat{y}_a + \sum_{b\in B} \hat{y}_b$$

The first inequality by feasibility and the second because M is a perfect matching.

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The first inequality by feasibility and the second because M is a perfect matching.

If all edges in M are tight equality holds and M is optimal.

## MWPM-BG: Primal dual algorithm

- The primal dual algorithm starts with a dual feasible solution, and a matching.
- At each time step it improves the number of tight edges and the weight of the matching, until the matching is perfect.
- At this point an optimal solution has been found.

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## MWPM-BG: Primal dual algorithm

function PRIMAL-DUAL MWPM-BG(A, B, E, w) $y_b = 0$ , for  $b \in B$  and  $y_a = \min_b w(a, b)$ , for  $a \in A$ E' = set of tight edges $M = \max$  cardinality matching in G' = (A, B, E')while *M* is not a perfect matching **do**  $\vec{E} = \{ e \in E' e \notin M \text{ (as} \overrightarrow{AB}) \} \cup \{ e \in M \text{ (as} \overrightarrow{BA}) \}$  $D = (A \cup B, \vec{E})$ % a directed graph.  $L = \{ v \in A \cup B \mid v \text{ is reachable in } D \text{ from an} \}$ unmatched vertex in A $\epsilon = \min_{a \in A \cap I} (w(a, b) - y_a - y_b)$  $v_a = v_a + \epsilon$ , for  $a \in A \cap L$  and  $v_b = v_b - \epsilon$ , for  $b \in B \cap L$ E' = set of tight edges  $M = \max$  cardinality matching in G' = (A, V, E')return M

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## MWPM-BG: Primal dual algorithm

#### Claim

After one iteration of the while loop

- y is a feasible dual solution.
- The number of tight edges strictly increases.

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## MWPM-BG: Primal dual algorithm

#### Claim

After one iteration of the while loop

- y is a feasible dual solution.
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#### Theorem

Algorithm PRIMAL-DUAL MWPM-BG terminates in  $O(|A \cup B|^3)$  iterations.

### Primal-Dual for vertex cover

#### $\mathbf{VC}$

Given a graph G = (V, E), we want to find a set S, with minimum number of vertices, so that every edge in G has at least one end point in S.

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#### $\mathbf{VC}$

Given a graph G = (V, E), we want to find a set S, with minimum number of vertices, so that every edge in G has at least one end point in S.

- $\bullet$  We know how to formulate VC as an IP problem
- We know how to relax the IP formulation as LP problem

## Primal-Dual for vertex cover

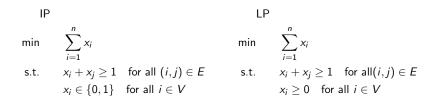
#### $\mathbf{VC}$

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- We know how to formulate VC as an IP problem
- We know how to relax the IP formulation as LP problem
- We know how to compute the dual of the LP problem

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### Vertex cover: LP relaxation

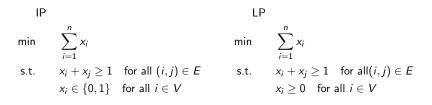


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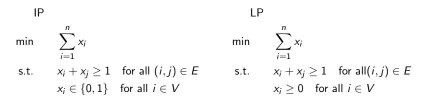
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• Let opt be the size of an optimal solution of the VC instance.

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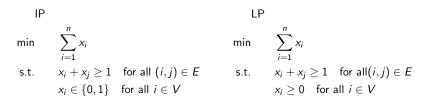


• Let opt be the size of an optimal solution of the VC instance.

• Let  $x^*$  be an optimal solution of the LP and  $s^* = \sum_{i=1}^n x_i^*$ .

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### Vertex cover: LP relaxation



Let opt be the size of an optimal solution of the VC instance.
Let x\* be an optimal solution of the LP and s\* = ∑<sub>i=1</sub><sup>n</sup> x<sub>i</sub>\*.
s\* ≤ opt

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### Vertex cover: Primal-Dual approximation

LP primal  
min 
$$\sum_{i=1}^{n} x_i$$
  
s.t.  $x_i + x_j \ge 1$   $e = (i, j) \in E$   
 $x_i \ge 0$   $i \in V$   
LP dual

 $\begin{array}{ll} \max & \sum_{e \in E} z_e \\ \text{s.t.} & \sum_{i \in e} z_e \leq 1 \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{array}$ 

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  - Set x<sub>i</sub> = 1. Freeze all the variables z<sub>e</sub> such that i ∈ e.

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• Cost of the solution? At the end of the algorithm x, z are feasible. Relaxed complementary slackness?.

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### Primal-Dual for weighted vertex cover

#### WVC

Given a vertex weighted graph G = (V, A, c) we want to find a set  $S \subset V$  with minimum weight, so that every edge in G has at least one end point in S.

- The problem is NP-hard and belongs to NPO.
- Can we formulate WVC as an IP problem?

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- Objective functon:  $\sum_{i=1}^{n} c_i x_i$ .
- Restrictions: for every edge  $(i,j) \in E$ ,  $x_i + x_j \ge 1$
- $x_i \in \{0, 1\}$
- The IP can be computed in polytime.

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#### Weighted vertex cover: LP relaxation

IP		LP	
min	$\sum_{i=1}^{n} c_i x_i$	min $\sum_{i=1}^{n} c_i x_i$	
s.t.	$x_i + x_j \ge 1$ for all $(i, j) \in E$	s.t. $x_i + x_j \ge 1$ for $all(i,j) \in E$	Ξ
	$x_i \in \{0,1\}$ for all $i \in V$	$x_i \geq 0$ for all $i \in V$	

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LP dual

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LP dual  
max  $\sum z_e$ 

s.t. 
$$\sum_{i \in e} z_e \leq c_i$$
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- Start with the integer infeasible primal solution x = 0, and the dual feasible solution z = 0.
- Repeat while some constraint in primal is unsatisfied:
  - Increase all (unfrozen) variables z<sub>e</sub> until some dual constraint becomes tight (say, for vertex i).
  - Set  $x_i = 1$ . Freeze all the variables  $z_e$  such that  $i \in e$ .

Matchings in bipartite graphs Approximating Vertex Cover Weighted Vertex Cover Pricing method

#### Weighted vertex cover: Primal-Dual approximation

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### Weighted vertex cover: Primal-Dual approximation

- When the process stops, we have increased the variables *z<sub>e</sub>* suitably.
- Some vertices *i* were chosen  $(x_i = 1)$
- This set S of vertices is our output and again is a vertex cover.
- Cost of the solution? *x*, *z* are feasible.

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- Cost of the solution? *x*, *z* are feasible. Relaxed complementary slackness conditions?

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# Primal-Dual approximation: relaxed complementary conditions

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• So, relaxed complementary conditions hold for *r* = 2 and we have a 2-approximation for WVC.

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### Primal-Dual approximation: generalizing the approach

AA-GEI: Approx, Param and Stream Linear Programming approximation: Primal Dual algorithms

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## Primal-Dual approximation: generalizing the approach

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## Primal-Dual approximation: generalizing the approach

- In the algorithm, we increased the (active) dual variables simultaneously.
- Trying to get the highest (the best) lower bound that we can get for the primal minimization objective.
   In general, this step can be implemented solving another LP program!
- We can also increase edge variables one by one. This leads to another primal-dual approximation algorithm **PRICING METHOD**

Matchings in bipartite graphs Approximating Vertex Cover Weighted Vertex Cover Pricing method

#### Pricing method: another view of Primal-Dual

Matchings in bipartite graphs Approximating Vertex Cover Weighted Vertex Cover Pricing method

### Pricing method: another view of Primal-Dual

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Matchings in bipartite graphs Approximating Vertex Cover Weighted Vertex Cover Pricing method

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- Prices  $z_e$  are fair if, for any vertex cover S,  $\sum_e z_e \le w(S)$ .
- A vertex is tight with respect to a pricing z if  $\sum_{i \in e} z_e = c_i$ .

### Pricing algorithm

Set prices and find vertex cover simultaneously.

```
function PRICING WVC(G, c)

S = \emptyset;

for e \in E do

z[e] = 0 % initial price is 0

while there is (i,j) \in E so that neither i nor j is tight do

select such an edge e = (i,j)

Increase z[e] until i or j became tight.

Add to S the vertex (vertices) that became tight.

return S
```

### Pricing algorithm

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PRICING WVC is a 2-approximation for WVC.

- Follows directly from primal-dual arguments.
- However, **PRICING WVC** is a greedy algorithm.
- No LP solver has been used!