Linear Programming approximation: Primal Dual algorithms

Maria Serna

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1. Primal-Dual algorithms

Primal dual algorithms

- Primal-Dual algorithms iterate obtaining primal/dual feasible solutions by increasing values of variables until a restriction is tight (fulfilled with equality).
- If at some point objective functions match, we have found an optimal solution.
- If at some point relaxed complementary slackness holds, for some r, we have found a r-approximate solution.

Bipartite graphs

MAXIMUM WEIGHT MATCHING IN BIPARTITE GRAPHS) (MWM-BG)

Given a bipartite graph $G = (A, B, E)$ and a weight function $w : E \to R$ find a matching of maximum weight where the weight of matching M is given by $w(M) = \sum_{e \in M} w(e)$.

MINIMUM WEIGHT PERFECT MATCHING ON BIPARTITE GRAPHS)(mWPM-BG)

Given a bipartite graph $G = (A, B, E)$ and a weight function $w : E \to R \cup \infty$ find a matching of maximum weight where the weight of matching M is given by $w(M) = \sum_{e \in M} w(e)$.

Is MWM-BG $\leq$ mWPM-BG? YES!, even for complete bipartite graphs with $|A| = |B|$!

ILP for MWPM-BG

$$\begin{aligned} \min \quad & \sum_{a\in A, b\in B} w(a,b) x_{a,b} \\ \text{s.t.} \quad & \sum_{b\in B} x_{a,b} = 1 \quad \forall\, a \in A \\ & \sum_{a\in A} x_{a,b} = 1 \quad \forall\, b \in B \\ & x_{a,b} \in \{0,1\} \quad \forall\, a \in A, b \in B \end{aligned}$$

In the LP relaxation, the last changes to $x_{a,b} \geq 0 \quad \forall\, a \in A, b \in B$

ILP for MWPM-BG: The dual of the relaxed LP

Primal

$$\min \sum_{a\in A, b\in B} w(a,b) x_{a,b}$$

$$\text{s.t.} \quad \sum_{b\in B} x_{a,b} = 1 \quad \forall\, a \in A$$

$$\sum_{a\in A} x_{a,b} = 1 \quad \forall\, b \in B$$

$$x_{a,b} \geq 0 \quad \forall\, a \in A, b \in B$$

The dual has a variable for each vertex y_a, y_b and the form

$$\max \sum_{a\in A} y_a + \sum_{b\in B} y_b$$

$$\text{s.t.} \quad y_a + y_b \leq w(a,b) \quad \forall\, a \in A, b \in B$$

$$y_a \geq 0 \quad \forall\, a \in A$$

$$y_b \geq 0 \quad \forall\, b \in B$$

ILP for MWPM-BG: tight edges

An edge $e = (a, b)$ is tight, for a dual feasible solution y, if $y_a + y_b = w(e)$.

Let $\hat{y}$ be dual-feasible, and let M be a perfect matching in $G = (A, B, E)$, then

$$w(M) = \sum_{(a,b)\in M} w(a, b) \geq \sum_{(a,b)\in M} (\hat{y}_a + \hat{y}_b) \geq \sum_{a\in A} \hat{y}_a + \sum_{b\in B} \hat{y}_b$$

The first inequality by feasibility and the second because M is a perfect matching.

If all edges in M are tight equality holds and M is optimal.

MWPM-BG: Primal dual algorithm

- The primal dual algorithm starts with a dual feasible solution, and a matching.
- At each time step it improves the number of tight edges and the weight of the matching, until the matching is perfect.
- At this point an optimal solution has been found.

MWPM-BG: Primal dual algorithm

function PRIMAL-DUAL MWPM-BG(A, B, E, w)
  $y_b = 0$, for $b \in B$ and $y_a = \min_b w(a, b)$, for $a \in A$
  E' = set of tight edges
  M = max cardinality matching in $G' = (A, B, E')$
  while M is not a perfect matching **do**
    $\overrightarrow{E} = \{e \in E' e \notin M\ (as \overrightarrow{AB})\} \cup \{e \in M\ (as \overrightarrow{BA})\}$
    $D = (A \cup B, \overrightarrow{E})$ % a directed graph.
    $L = \{v \in A \cup B \mid v$ is reachable in D from an unmatched vertex in $A\}$
    $\epsilon = \min_{a \in A \cap L}(w(a, b) - y_a - y_b)$
    $y_a = y_a + \epsilon$, for $a \in A \cap L$ and $y_b = y_b - \epsilon$, for $b \in B \cap L$
    E' = set of tight edges
    M = max cardinality matching in $G' = (A, V, E')$
  return M

MWPM-BG: Primal dual algorithm

Claim

After one iteration of the while loop

- y is a feasible dual solution.
- The number of tight edges strictly increases.

Theorem

Algorithm PRIMAL-DUAL MWPM-BG *terminates in* $O(|A \cup B|^3)$ *iterations.*

Primal-Dual for vertex cover

VC

Given a graph $G = (V, E)$, we want to find a set S, with minimum number of vertices, so that every edge in G has at least one end point in S.

- We know how to formulate VC as an IP problem
- We know how to relax the IP formulation as LP problem
- We know how to compute the dual of the LP problem

Vertex cover: LP relaxation

IP

$$\begin{array}{lll} \min & \sum_{i=1}^{n} x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} & \text{for all } i \in V \end{array}$$

LP

$$\begin{array}{lll} \min & \sum_{i=1}^{n} x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & \text{for all} (i,j) \in E \\ & x_i \geq 0 & \text{for all } i \in V \end{array}$$

- Let opt be the size of an optimal solution of the VC instance.
- Let x^* be an optimal solution of the LP and $s^* = \sum_{i=1}^{n} x_i^*$.
- $s^* \leq$ opt

Vertex cover: Primal-Dual approximation

LP primal

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad e = (i,j) \in E \\ & x_i \geq 0 \quad i \in V \end{aligned}$$

LP dual

$$\begin{aligned} \max \quad & \sum_{e \in E} z_e \\ \text{s.t.} \quad & \sum_{i \in e} z_e \leq 1 \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{aligned}$$

- Start with the integer infeasible primal solution $x = 0$, and the dual feasible solution $z = 0$.
- Repeat while some constraint in primal is unsatisfied:
  - Increase all (unfrozen) variables z_e until some dual constraint becomes tight (say, for vertex i).
  - Set $x_i = 1$. Freeze all the variables z_e such that $i \in e$.

Vertex cover: Primal-Dual approximation

- When the process stops, we have increased the variables z_e suitably.
- Some vertices i were chosen ($x_i = 1$)
- This set S of vertices is our output.
- Is S a vertex cover?
  Otherwise, we would have continued as some primal constraint were still unsatisfied.
- Cost of the solution?
  At the end of the algorithm x, z are feasible. Relaxed complementary slackness?.

Primal-Dual approximation: relaxed complementary conditions

Two conditions hold:

- Primal:
  If $x_i > 0$, we have frozen $x_i = 1$ at some step,
  then $\sum_{i \in e} z_e = 1$.
- Dual:
  If $z_e > 0$, z_e is increased. We do not know if it is because for one or both endpoints the constraints got tight, but
  $x_i + x_j \leq 2 \leq 2c_i$, for $e = (i, j)$.
- So, relaxed complementary slackness conditions hold for $r = 2$. A 2-approximation ☺.

Primal-Dual for weighted vertex cover

WVC

Given a vertex weighted graph $G = (V, A, c)$ we want to find a set $S \subset V$ with minimum weight, so that every edge in G has at least one end point in S.

- The problem is NP-hard and belongs to NPO.
- Can we formulate WVC as an IP problem?
- Variables: $x_1 \dots x_n$, $x_i = 1$ iff $i \in S$.
- Objective functon: $\sum_{i=1}^{n} c_i x_i$.
- Restrictions: for every edge $(i, j) \in E$, $x_i + x_j \geq 1$
- $x_i \in \{0, 1\}$
- The IP can be computed in polytime.

Weighted vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{aligned}$$

Weighted vertex cover: Primal-Dual approximation

LP primal

$$\min \quad \sum_{i=1}^{n} c_i x_i$$
$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \text{for all}(i,j) \in E$$
$$x_i \geq 0 \quad \text{for all } i \in V$$

LP dual

$$\max \quad \sum_{e \in E} z_e$$
$$\text{s.t.} \quad \sum_{i \in e} z_e \leq c_i \quad \text{for all } i \in V$$
$$z_e \geq 0 \quad \text{for all } e \in E$$

- Start with the integer infeasible primal solution $x = 0$, and the dual feasible solution $z = 0$.
- Repeat while some constraint in primal is unsatisfied:
  - Increase all (unfrozen) variables z_e until some dual constraint becomes tight (say, for vertex i).
  - Set $x_i = 1$. Freeze all the variables z_e such that $i \in e$.

Weighted vertex cover: Primal-Dual approximation

- When the process stops, we have increased the variables z_e suitably.
- Some vertices i were chosen ($x_i = 1$)
- This set S of vertices is our output and again is a vertex cover.
- Cost of the solution? x, z are feasible. Relaxed complementary slackness conditions?

Primal-Dual approximation: relaxed complementary conditions

Two conditions hold:

- Primal:
  If $x_i > 0$, we have frozen $x_i = 1$ at some step,
  then $\sum_{i \in e} z_e = c_i$.
- Dual:
  If $z_e > 0$, z_e is increased. We do not know if it is because for one or both endpoints the constraints got tight, but
  $x_i + x_j \leq 2 \leq 2c_i$, for $e = (i, j)$.
- So, relaxed complementary conditions hold for $r = 2$ and we have a 2-approximation for WVC.

Primal-Dual approximation: generalizing the approach

- In the algorithm, we increased the (active) dual variables simultaneously.
- Trying to get the highest (the best) lower bound that we can get for the primal minimization objective.
  In general, this step can be implemented solving another LP program!
- We can also increase edge variables one by one. This leads to another primal-dual approximation algorithm PRICING METHOD

Pricing method: another view of Primal-Dual

- Each edge must be covered by some vertex.
- Edge $e = (i, j)$ pays price $z_e \geq 0$ to use both vertex i and j.
- Fairness: Edges incident to vertex i should pay $\leq c_i$ in total.
- Prices z_e are fair if, for any vertex cover S, $\sum_e z_e \leq w(S)$.

- A vertex is tight with respect to a pricing z if $\sum_{i \in e} z_e = c_i$.

Pricing algorithm

Set prices and find vertex cover simultaneously.

function PRICING WVC(G, c)
  $S = \emptyset$;
  for $e \in E$ **do**
    $z[e] = 0$ % initial price is 0
  while there is $(i, j) \in E$ so that neither i nor j is tight **do**
    select such an edge $e = (i, j)$
    Increase $z[e]$ until i or j became tight.
    Add to S the vertex (vertices) that became tight.
  return S

Pricing algorithm

Theorem

PRICING WVC *is a 2-approximation for* WVC*.*

- Follows directly from primal-dual arguments.
- However, PRICING WVC is a greedy algorithm.
- No LP solver has been used!