Linear Programming approximation: Primal Dual

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Spring 2024

1. LP duality
2. Primal-Dual algorithms

Primal Dual

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- Many real-life problems can be modeled as Integer Linear Programs (IP).
- Since IPs are NP-hard to solve, they are often relaxed to a linear program (shortened as LP).
- Modus operandi: solve the linear program in polynomial time, and extract useful information about an integer optimum solution.
- However, for certain problems, we do not need to even solve the LP to get good (reasonable approximation factor or even optimal) solutions to our problem using duality to control improvements.

History

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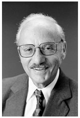
- George Dantzig started linear programming (1947) , and his ideas contain the first germs of primal dual algorithms. The Hungarian method was an application of the paradigm.
- Jack R. Edmonds gave the first (sophisticated) application of the paradigm in his work on maximum weight matchings in arbitrary graphs (1965).
- Bar-Yehuda and Even first enunciated the paradigm in their work on the weighted Vertex Cover problem (1981).

Dantzig  Edmonds 

Primal, Dual and Weak Duality

Consider a LP in n variables $x = (x_1, \dots, x_n)$ with m constraints represented by matrix A, independent terms b, and objective function b.

Primal

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

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The dual is an effort to construct the best lower bound for the primal objective function.

Searching for a lower bound: The best one?

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$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array}$$

DUAL

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- Working from the dual trying to get the best lower bound we come back to the primal.
- Another example that you know is the pair MaxFlow-MinCut if you write the LP formulation of MaxFlow you can check that the dual is a LP formulation for MinCut

Strong and Weak duality theorem

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If the primal has an optimal solution x^* then the dual has an optimal solution y^* such that $c^T x^* = b^T y^*$

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Theorem (Weak Duality)

For every feasible solution x to the primal and every solution z to the dual,

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j z_j$$

Conditions for optimality: Complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal complementary slackness

If $x_i > 0$, then $\sum_{j=1}^{m} a_{ij} z_j = c_i$.

Dual complementary slackness

If $z_j > 0$, then $\sum_{i=1}^{n} a_{ij} x_i = b_j$.

Conditions for optimality: Complementary slackness

Theorem

If (x, y) satisfies complementary slackness, then x and y are optimal solutions for primal and dual problems, respectively.

Proof.

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij} z_j) x_i = \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij} x_i) z_j = \sum_{j=1}^{m} b_j z_j$$

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Relaxed complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal relaxed complementary slackness

If $x_i > 0$, then $\sum_{j=1}^{m} a_{ij} z_j \geq c_i / \alpha$.

Dual relaxed complementary slackness

If $z_j > 0$, then $\sum_{i=1}^{n} a_{ij} x_i \leq \beta\, b_j$.

for some factors $\alpha, \beta \geq 1$

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If x is integral and primal and dual relaxed complementary slackness hold?

Relaxed complementary slackness

Theorem

Let Π be a minimization integer program and Π-LP its LP-relaxation. Suppose a primal (integer) feasible solution x of Π and a dual feasible solution y of Π-LP satisfy the primal-dual relaxed complementary slackness, for some $\alpha, \beta > 1$, and x is integral, then x is a $\alpha\beta$-approximation.

Relaxed complementary slackness

Proof.

$$\sum_{i=1}^{n} c_i x_i \leq \alpha \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij} z_j) x_i = \alpha \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij} x_i) z_j \leq \alpha\beta \sum_{j=1}^{m} b_j z_j$$

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By weak duality $\sum_{j=1}^{m} b_j z_j \leq \sum_{i=1}^{n} c_i x'_i$ for any feasible x', in particular for the optimal solution of the IP, therefore

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By weak duality $\sum_{j=1}^{m} b_j z_j \leq \sum_{i=1}^{n} c_i x_i'$ for any feasible x', in particular for the optimal solution of the IP, therefore

$$\sum_{i=1}^{n} c_i x_i \leq \alpha\beta \sum_{j=1}^{m} b_j z_j \leq \alpha\beta \, \text{opt}$$

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