Approximation algorithms: Linear and Integer Programming

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  - satisfy the set of linear inequalities (equations or constraints),
  - maximize or minimize the objective function.
- LP is a pure algebraic problem.

Linear programming: An example

$$
\begin{aligned}
\max\ & x_1 + 6x_2 \\
\text{subject to}\ & \\
& x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

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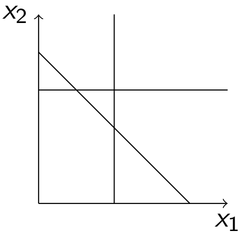

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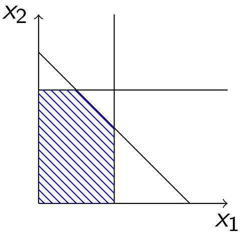

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    For ex. $x \geq 2$ and $x \leq 1$
  - The constrains are so loose that the feasible region is unbounded allowing the objective function to go to ∞ .
    For ex. max $x_1 + x_2$ subject to $x_1, x_2 \geq 0$

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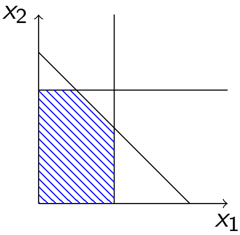
- In a feasible linear programming the optimum is achieved at a vertex of the feasible region.



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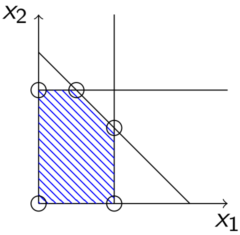

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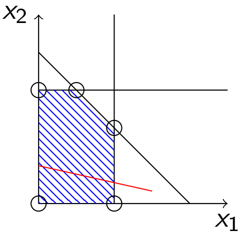

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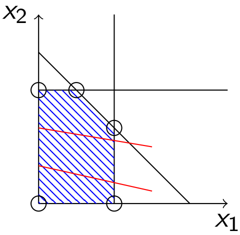

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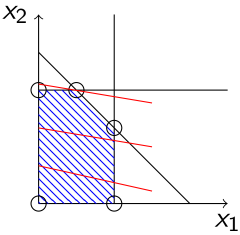

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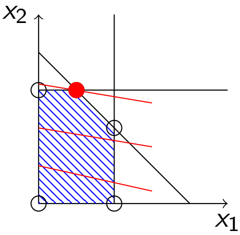

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- From $=$ to $\leq$ (or to $\geq$)
  put two versions one with $\leq$ and the other with $\geq$, multiply the last one by -1.
- From x unrestricted to non-negative variables,
  create two new variables x^+ and x^-, both non negative, replace x by $x^+ - x^-$.

Linear programming: standard formulation

LP standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Where

- $x = (x_1, \ldots, x_n)$, $c = (c_1, \ldots, c_n)$.
- $b^T = (b_1, \ldots, b_m)$
- A is a $n \times m$ matrix.

Linear programming: problem

Given

- $c = (c_1, \ldots, c_n)$,
- $b^T = (b_1, \ldots, b_m)$,
- and a $n \times m$ matrix A.

find $x = (x_1, \ldots, x_n) \geq 0$, so that

- $Ax \geq b$ and $c^T x$ is minimized.

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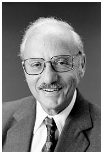

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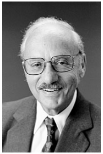

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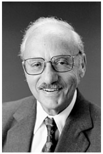

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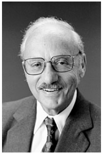

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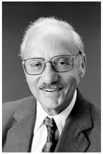

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  (exponential time Klee and Minty 1972)
- Ellipsoid method: Khachiyan 1979 ($O(n^6)$)
- Interior-point method: Karmarkar 1984 ($O(n^3)$)
- Most used algorithm is still Simplex (fast on average).
- Many commercial LP solvers CPLEX and open source Gurobi

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- Many NPO problems can be easily expressed as IP or MIP problems
- IP is NP-hard

Max SAT as integer program

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  The variables will be restricted to have values in $\{0, 1\}$
  This is a simplification of saying that they must hold integer values and that all of them are ≤ 1.

Max SAT as integer program

Max SAT-IP

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{m} y_j \\
\text{s.t.} \quad & \sum_{i \in P(j)} x_i + \sum_{i \in N(j)} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m \\
& y_j \in \{0, 1\} \qquad 1 \leq j \leq m \\
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The size of the IP is polynomial in the size of the Max SAT, so the transformation is a polynomial Turing reduction from Max SAT to IP.

Vertex cover as integer program

VC

Given a graph $G = (V, E)$ we want to find a set $S \subset V$ with minimum cardinality, so that every edge in G has at least one end point in S.

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VC-IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i, j) \in E \\ & x_i \in \{0, 1\} \quad \text{for all } i \in V \end{aligned}$$

Weighted Vertex cover as integer program

WVC

Given a graph $G = (V, A)$ with weights w associated to the vertices, we want to find a set $S \subset V$ with minimum weight, so that every edge in G has at least one end point in S.

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Exercise

Try to write a LP or IP formulation for the problems

- Min Weighted Matching
- Set cover
- Max Flow

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- The IP can be relaxed to a linear program (LP) by eliminating the integrity constraints.
- By doing so the optimum cost can only improve, i.e., opt of LP is better than opt of IP.
- We can solve the LP in polynomial time.
- The LP optimal solution might not be integral, when possible, transform it to get a feasible integer solution not far from opt of IP.

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VC-LP

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Lemma

VC-LP *has an optimal solution* x^* *such that* $x_i \in \{0, 1, 1/2\}$. *Furthermore, such a solution can be computed in polynomial time.*

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$$y_i' = \begin{cases} y_i - \epsilon & 0 < y_i < 1/2 \\ y_i + \epsilon & 1/2 < y_i < 1 \\ y_i & \text{otherwise} \end{cases} \qquad y_i'' = \begin{cases} y_i + \epsilon & 0 < y_i < 1/2 \\ y_i - \epsilon & 1/2 < y_i < 1 \\ y_i & \text{otherwise} \end{cases}$$

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$\sum y_i = (\sum y'_i + \sum y''_i)/2$, so both are optimal solutions.

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Let x defined as $x_i = 0$ if $y'_i = 0$, $x_i = 1$ otherwise.

return (x)

Vertex cover

function $\textsc{Relax+Round VC}(G)$
  Construct the LP-VC associated G
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$\textsc{Relax+Round VC}$

Vertex cover

function Relax+Round VC(G)
Construct the LP-VC associated G
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- is a 2-approximation for VC.

Weighted vertex cover: Relax+Round approximation

LP WVC

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{aligned}$$

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Minimum 2-Satisfiability

MIN 2-SAT

Given a Boolean formula in 2-CNF, determine whether it is satisfiable and, in such a case, find a satisfying assignment with minimum number of true variables.

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- MIN 2-SAT IP formulation?

Minimum 2-Satisfiability: IP formulation

Suppose that F has n variables $x_1, \ldots x_n$ and m clauses with 2 literals per clause

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IP Min 2-SAT

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\begin{array}{lrl}
 & \min \quad \displaystyle\sum_{i=1}^{n} x_i & \\
\text{s.t.} & x_i + x_j \geq 1 & \text{for all clauses } (x_i \vee x_j) \in F \\
 & (1 - x_i) + x_j \geq 1 & \text{for all clauses } (\overline{x}_i \vee x_j) \in F \\
 & (1 - x_i) + (1 - x_j) \geq 1 & \text{for all clauses } (\overline{x}_i \vee \overline{x}_j) \in F \\
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LP Min 2-SAT is obtaining replacing $x_i \in \{0, 1\}$ by $x_i \geq 0$.

Minimum 2-Satisfiability: LP relaxation

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- When $y_i = 1/2$?

Minimum 2-Satisfiability: LP relaxation

- Let y be an optimal solution to IP Min 2-SAT.
- What to do when $y_i = 1/2$? 1? 0?

Minimum 2-Satisfiability: LP relaxation

- Let y be an optimal solution to IP Min 2-SAT.
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- If F contains the clauses $(x_i \vee x_j)$ and $(\overline{x}_i \vee \overline{x}_j)$ and $y_i = y_j = 1/2$, neither $x_i = x_j = 1$ nor $x_i = x_j = 0$ satisfy the formula.

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- Rounding those values to 1 or 0 would keep the approximation ratio to 2, provided the constructed solution x to MIN 2-SAT is still a satisfying assignment.

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- Let y be an optimal solution to IP Min 2-SAT.
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- $F_1 =$ clauses whose two variables have y value $= 1/2$.
- Rounding those values to 1 or 0 would keep the approximation ratio to 2, provided the constructed solution x to Min 2-SAT is still a satisfying assignment.
- Any satisfying assignment for the clauses in F_1 and get a 2-approximation ☺

Minimum 2-Satisfiability: Relax+Round approximation

function RELAX+ROUND MIN 2-SAT(F)
  if F is not satisfiable **then return** false
  Construct the LP Min 2-SAT, I
  $y = LP.solve(I)$
  for $i = 1, \dots, n$ **do**
    if $y_i' < 1/2$ **then** $x_i = 0$
    if $y_i' > 1/2$ **then** $x_i = 1$
  F_1 = clauses with both y values $= 1/2$.
  Let $J = \{j \mid x_j \in F_1\}$
  for i=1,. . . , n **do**
    if $y_i = 1/2$ and $i \notin J$ **then** $x_i = 1$
  Complete x with a satisfying assignment for F_1
  return (x)

Minimum 2-Satisfiability: Relax+Round approximation

Theorem

RELAX+ROUND MIN 2-SAT *is a 2-approximation for* MIN 2-SAT.

Max Satisfiability

MAX SAT

Given a Boolean formula in CNF and weights for each clause, find a Boolean assignment to maximize the weight of the satisfied clauses.

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Suppose that F has n variables $x_1, \dots x_n$ and m clauses $C_1, \dots, C_m$.

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IP Max SAT

$$\begin{aligned}
\max \quad & \sum_{j=1}^{m} w_j z_j \\
\text{s.t.} \quad & \sum_{x_i \in C_j} y_i + \sum_{\overline{x}_i \in C_j} (1 - y_i) \geq z_j \quad \jmath = 1, \ldots, m \\
& y_i \in \{0,1\} \qquad 1 \leq i \leq n \\
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LP Max SAT is obtaining replacing $a \in \{0,1\}$ by $0 \leq a \leq 1$.

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- The optimal LP solution is used as an indicator of the probability that the variable has to been set to 1.

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- The optimal LP solution is used as an indicator of the probability that the variable has to been set to 1.
- The performance of a randomized algorithm is the expected number of satisfiable clause.
- This expectation has to be compared with opt.

Max Satisfiability: Relax+RRound

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- Let (y^*, z^*) be an optimal solution of LP Max SAT
- Let Z_j be the indicator random variable for the event that clause C_j is satisfied.
- Assume that C_j has k-literals and that ℓ of them are negated variables.

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Lemma

For any $1 \leq j \leq m$, $E[Z_j] \geq z_j^*(1 - 1/e)$.

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Lemma

For any $1 \leq j \leq m$, $E[Z_j] \geq z_j^*(1 - 1/e)$.

Recall $(a_1 \dots a_k)^{1/k} \leq (a_1 + \dots + a_k)/k$

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For any $1 \leq j \leq m$, $E[Z_j] \geq z_j^*(1 - 1/e)$.

Recall $(a_1 \dots a_k)^{1/k} \leq (a_1 + \cdots + a_k)/k$ or equivalently $(a_1 \dots a_k) \leq ((a_1 + \cdots + a_k)/k)^k$

Max Satisfiability: Relax+RRound

Proof.

☺

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Z_j is an indicator random variable, and so
$E[Z_j] = Pr[Z_j = 1] = 1 - Pr[Z_j = 0]$

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$$Pr[Z_j = 0] = \prod_{x_i \in C_j} (1 - y_i^*) \cdot \prod_{\overline{x}_i \in C_j} y_i^* \le \left(\frac{(k - \ell) - \sum_{x_i \in C_j} y_i^* + \sum_{\overline{x}_i \in C_j} y_i^*}{k} \right)^k$$

$$\le \left(\frac{(k - \sum_{x_i \in C_j} y_i^* - \sum_{\overline{x}_i \in C_j} (1 - y_i^*))}{k} \right)^k \le \left(\frac{(k - z_j^*)}{k} \right)^k \le \left(1 - \frac{z_j^*}{k} \right)^k$$

$$E[Z_j] \ge 1 - \left(1 - \frac{z_j^*}{k} \right)^k \ge z_j^* \left(1 - \frac{1}{k} \right)^k \ge z_j^* (1 - 1/e)$$

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$$\leq \left(\frac{(k - \sum_{x_i \in C_j} y_i^* - \sum_{\overline{x}_i \in C_j} (1 - y_i^*))}{k} \right)^k \leq \left(\frac{(k - z_j^*)}{k} \right)^k \leq \left(1 - \frac{z_j^*}{k} \right)^k$$

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☺

Max Satisfiability: Relax+RRound approximation

Theorem

RELAX+RROUND *is a* $e/(e-1)$*-approximation for* MAX SAT.

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Proof.

- Let (y^*, z^*) be an optimal solution of LP Max SAT
- Let Z_j be the indicator r.v.a for clause C_j is satisfied.
- Let W be the r.v. weight of satisfied clauses: $W = \sum_{j=1}^{m} w_j Z_j$.

Max Satisfiability: Relax+RRound approximation

Theorem

RELAX+RROUND *is a* $e/(e-1)$*-approximation for* MAX SAT.

Proof.

- Let (y^*, z^*) be an optimal solution of LP Max SAT
- Let Z_j be the indicator r.v.a for clause C_j is satisfied.
- Let W be the r.v. weight of satisfied clauses: $W = \sum_{j=1}^{m} w_j Z_j$.
- $E[W] = \sum_{j=1}^{m} w_j E[Z_j] \geq (1 - 1/e) \sum_{j=1}^{m} w_j z_j^* \geq (1 - 1/e)\text{opt}$

☺

Max Satisfiability:RandAssign

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function RANDASSIGN(F)
  for i=1,. . . , n **do**
    Set $x_i = 1$ with probability $1/2$
  return (x)

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Theorem

RANDASSIGN *is a 2-approximation for* MAX SAT.

Max Satisfiability:RandAssign

function RANDASSIGN(F)
  for i=1,. . . , n **do**
    Set $x_i = 1$ with probability $1/2$
  return (x)

Theorem

RANDASSIGN *is a 2-approximation for* MAX SAT.

Proof.

$E[W] = \sum_{j=1}^{m} w_j E[Z_j] = \sum_{j=1}^{m} w_j \left(1 - \left(\frac{1}{2}\right)^{k_j}\right) \geq \frac{1}{2} \sum_{j=1}^{m} w_j \geq \frac{1}{2}\text{opt}.$ ☺

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We move from $r = 2$ (RANDASSIGN) to $r = 1.581977$ (RELAX+RROUND).

Max Satisfiability:Best2

function $\textsc{Best2}(F)$
$x_1, W_1 = \textsc{RandAssign}(F)$
$x_2, W_2 = \textsc{Relax+RRound}(F)$
if $W_1 \geq W_2$ **then**
return (x_1)
else
return (x_2)

Max Satisfiability:Best2

function $\textsc{Best2}(F)$
  $x_1, W_1 = \textsc{RandAssign}(F)$
  $x_2, W_2 = \textsc{Relax+RRound}(F)$
  if $W_1 \geq W_2$ **then**
    return (x_1)
  else
    return (x_2)

Theorem

$\textsc{Best2}$ *is a* $4/3$ *(1.33333)-approximation for* $\textsc{Max SAT}$.

Max Satisfiability:Best2

Max Satisfiability:Best2

Proof.

Max Satisfiability:Best2

Proof.

- $E[W] = E[\max\{W_1, W_2\}] \geq E[(W_1 + W_2)/2\}]$.

Max Satisfiability:Best2

Proof.

- $E[W] = E[\max\{W_1, W_2\}] \geq E[(W_1 + W_2)/2\}]$.

$$E[W] \geq \sum_{j=1}^{m} w_j \left[\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{k_j}\right) + \frac{1}{2} z_j^* \left(1 - \left(\frac{1}{k_j}\right)^{k_j}\right)\right]$$

$$\geq \sum_{j=1}^{m} w_j \frac{3}{4} z_j^* \geq \frac{3}{4} \sum_{j=1}^{m} w_j z_j^* \geq \frac{3}{4}\text{opt}.$$

Max Satisfiability:Best2

Proof.

- Is $\left[\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{k_j}\right)+\frac{1}{2}z_j^*\left(1-\left(\frac{1}{k_j}\right)^{k_j}\right)\right] \geq \frac{3}{4}z_j^*$?

Max Satisfiability:Best2

Proof.

- Is $\left[\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{k_j}\right)+\frac{1}{2}z_j^*\left(1-\left(\frac{1}{k_j}\right)^{k_j}\right)\right]\geq\frac{3}{4}z_j^*$?
- $k_j=1$: $\frac{1}{2}\frac{1}{2}+\frac{1}{2}z_j^*\geq\frac{3}{4}z_j^*$.

Max Satisfiability:Best2

Proof.

- Is $\left[\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{k_j}\right)+\frac{1}{2}z_j^*\left(1-\left(\frac{1}{k_j}\right)^{k_j}\right)\right] \geq \frac{3}{4}z_j^*$?
- $k_j = 1$: $\frac{1}{2}\frac{1}{2}+\frac{1}{2}z_j^* \geq \frac{3}{4}z_j^*$.
- $k_j = 2$: $\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{3}{4}z_j^* \geq \frac{3}{4}z_j^*$.

Max Satisfiability:Best2

Proof.

- Is $\left[\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{k_j}\right)+\frac{1}{2}z_j^*\left(1-\left(\frac{1}{k_j}\right)^{k_j}\right)\right]\geq\frac{3}{4}z_j^*$?
- $k_j=1$: $\frac{1}{2}\frac{1}{2}+\frac{1}{2}z_j^*\geq\frac{3}{4}z_j^*$.
- $k_j=2$: $\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{3}{4}z_j^*\geq\frac{3}{4}z_j^*$.
- $k_j\geq 3$: the minimum possible of each term is

$$\frac{1}{2}\frac{7}{8}+\frac{1}{2}\left(1-\frac{1}{e}\right)z_j^*\geq\frac{3}{4}z_j^*$$

Max Satisfiability:Best2

Proof.

- Is $\left[\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{k_j}\right)+\frac{1}{2}z_j^*\left(1-\left(\frac{1}{k_j}\right)^{k_j}\right)\right] \geq \frac{3}{4}z_j^*$?
- $k_j = 1$: $\frac{1}{2}\frac{1}{2}+\frac{1}{2}z_j^* \geq \frac{3}{4}z_j^*$.
- $k_j = 2$: $\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{3}{4}z_j^* \geq \frac{3}{4}z_j^*$.
- $k_j \geq 3$: the minimum possible of each term is

$$\frac{1}{2}\frac{7}{8}+\frac{1}{2}\left(1-\frac{1}{e}\right)z_j^* \geq \frac{3}{4}z_j^*$$

☺

1. LP and IP
2. Relax and round
3. LP Duality