Algorithms for approximation, parameterization, and data streams

Maria Serna

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Not necessarily in this order.

References:Approximation

(S. = Springer)

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(S. = Springer)

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References: Data Streaming

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  data stream problem

Decision Problems: Complexity classes

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- NP non-deterministic polynomial time
  Syntactic definition!

You already known about these classes!

Optimization Problems

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That is the function problem whose goal, with respect to an instance x, is to find an optimum solution, that is, a feasible solution y such that

$$y = \mathrm{goal}\{(m(x, y') \mid y' \in \mathrm{sol}(x)\}.$$

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  Syntactic definition (next slide)

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  Given $x \in I$ and an integer k
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- A NPO problem is NP-hard if its bounded version is NP-complete.

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- Can an NPO problem be NP-complete?

Some NPO problems

MIN-BIN PACKING

Given n objects, object i has volume v_i, $0 \leq v_i \leq 1$, compute the minimum number of unit bins needed to pack all the objects.

MAX-SAT

Given a CNF formula F, compute an assignment that satisfies the maximum number of clauses.

MAX-W-SAT

Given a CNF formula F, in which each clause has an assigned weight. Define the value of an assignment as the sum of the weights of the satisfied clauses. The problem consists in computing an assignment with maximum value.

And the bounded literal per clause families MAX-K-SAT.

Some NPO problems: hardness

Which problems in the previous slide are NP-hard? Why?

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- Approximation algorithm: a feasible solution with a performance guarantee.
- Parameterization: solve efficiently some slices of the problem.

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$\mathcal{P}$ is r-approximable in polynomial time if there is a polynomial time computable r-approximation for $\mathcal{P}$.

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- Why $r \geq 1$?
- In which cases we can have $r = 1$?
- How would you like r to be?
- Is there any trivial condition about r, for maximization problems? for minimization ones?

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- **FPTAS** Fully Polynomial Time Approximation Schema
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Negative results through hardness APX-hard etc.

Hard to approximate problems

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Non-approximable

unless P =NP, for any r at most a polynomial function of n, there is no polynomial time r-approximation algorithm

First approximation algorithms

- We start analyzing approximation algorithms based in the greedy technique.
- One algorithm for MIN-BIN PACKING
- And two algorithms for a load-balancing problem.

Nex Fit: An approximation algorithm for BinPacking

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Next Fit places the n objects, one after the other, as follows:

- Opens B_1 and places the first object in B_1.
- If the i-th object fits in the open box, we put it inside. Otherwise, we close the bin, open the next one and place the object in it.

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For the case of $v_1 = 0.3$, $v_2 = 0.8$, and $v_3 = 0.7$, Next Fit solution needs three bins. But there is a solution with uses only two bins. Not optimal!

Nex Fit: An approximation algorithm for BinPacking

Theorem

Let x be an input to the MIN-BIN PACKING *problem and let $opt(x)$ be the minimum number of bins needed to pack de objects in x. If $NF(x)$ is the number of bins in the solution computed by Next Fit, then $opt(x) \leq NF(x) \leq 2opt(x)$.*

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Proof.

The first inequality is always true:

- MIN-BIN PACKING is a minimization problem
- Next Fit provides a feasible solution

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Let $V = \sum_{i=1}^{n} v_i$, we have $\text{opt}(x) \geq \lceil V \rceil$.

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Let us look to two consecutive bins in the Next Fit solution The total packet size in the two bins must be bigger than 1, otherwise we will never have opened the second bin. So,

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☺

Load Balancing problem

Processing scenario

- We have m identical machines;
  n jobs, job j has processing time t_j
- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.
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- Let $J(i)$ be the subset of jobs assigned to machine i.
- The load of machine i is $L_i = \sum_{j \in J(i)} t_j$
- The makespan is the maximum load on any machine, $L = \max_i L_i$.

Load Balancing problem

LBAL

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  Load balancing is hard even if $m = 2$ machines (reduction from Partition).

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- The problem is NP-hard and belongs to NPO.
  Load balancing is hard even if $m = 2$ machines (reduction from Partition).
- The approximation algorithm we propose is a greedy algorithm called list-scheduling.

List scheduling

List scheduling

For $j = 1, \dots, n$:

Assign job j to the machine having smallest load so far.

List scheduling: Implementation

function LIST SCHEDULING(m, n, T)
  for $i = 1, dots, m$ **do**
    $L[i] = 0$ — load on machine
    $J[i] = \emptyset$ — jobs assigned to
  end for
  for $j = 1, \ldots, n$ **do**
    $i = \text{argmin}_k L_k$ — machine with smallest load
    $J[i] = J[i] \cup \{j\}$
    $L[i] = L[i] + T[j]$
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Cost: Using a priority queue to maintain L, the cost is $O(n \log m)$

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- $L^* \geq \max_j t_j$ and $L^* \geq \frac{1}{m} \sum_j t_j$.

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- $L^* \geq \max_j t_j$ and $L^* \geq \frac{1}{m} \sum_j t_j$.
- Assume that $L = L_i$. Let j be the last job scheduled in machine i.
- When job j was assigned, all the other machines have higher load, so $L_i - t_j \leq L_k$, for $k \neq i$.

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☺

Approximation rate: tightness

- When we design an approximation algorithm, we wish to approach the best possible approximation ratio.
- Which one is the best for a problem, as usual, requires some complexity consideration and, in some cases, we have answers like
  "this problem cannot be approximated for $r \leq \ldots$
  unless P $=$NP"

Approximation algorithm: tightness

- We can ask a similar tightness question, not for the optimization problem, but about the approximation algorithm at hand.
- In this case the question is about the tightness in the analysis of the approximation ratio. The value of r is correct or can it be reduced further?
- We can show the tightness in the analysis of r by finding an input x so that the rate between $\text{opt}(x)$ and $\mathcal{A}(x)$ rules out any improvement on r.
- The previous step involves computing the optimal solution for a particularly adequate input, not solving the optimization problem.

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- The approximation rate is tight ☺

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- The 3/2 bound on the approximation rate is not tight
- In fact LONGEST PROCESSING FIRST is a 4/3-approximation algorithm [Graham 1969]
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- $4/3$ is tight:
  - m machines
  - $n = 2m + 1$ jobs
  - 2 jobs of length $m, m+1, \ldots, 2m-1$ and one more job of lenght m.
  - $L^* = 3m$ and $L = 4m - 1$, which gives a ratio tending to $4/3$.

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Max Cut

MAX-CUT

Given a graph $G = (V, A)$ we want to find a partition of V into V_1, V_2 in such a way that

$$\mathrm{cut}(V_1, V_2) = \|\{(u, v) \mid u \in V_1\ v \in V_2\}\|$$

is maximum.

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- The problem is NP-hard and belongs to NPO.
- Let us analyze a local search algorithm using the HillClimbing paradigm.

Local search

Local search

- A neighborhood structure is defined on the set of solutions.
- The algorithm performs an exploration of the neighborhood graph.
- Hill Climbing: It starts at one feasible solution and moves to a better one. It finishes at a local optimum, when no neighbor improves the value of m.
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A local optimum is a solution such that all its neighbors have equal or worse cost.

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Using $\mathcal{N}(G)$ we consider the following algorithm:

HillClimbing Max-Cut (G:graph, n: integer)
V_1, V_2: set of $[1 \ldots n]$;
$V_1 := \varnothing$; $V_2 :=$ V(G);
while not local-optimum(V_1, V_2) do
$(V_1, V2)$:= a neighboring partition of (V_1, V_2) with improved cost

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- At each step the cut is increased in one unit.

HillClimbing MaxCut

Lemma

Let $G = (V, A)$ be a graph, if (V_1, V_2) is a local optimum of $\mathcal{N}(G)$ then $cut((V_1, V_2)) \leq 2opt(G)$.

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Theorem

HILLCLIMBING MAX-CUT *is a polynomial 2-approximation algorithm for* MAX-CUT.

The class PLS

- Polynomial Local Search (PLS) is a complexity class that models the difficulty of finding a locally optimal solution to an optimization problem.

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- PLS problems always have a solution!

The class PLS

- The conditions guarantee that a navigation step can be performed in polynomial time.
- The size of the solution set do not guarantee that an exploration will end within polynomial time.
- However the computation uses only polynomial space.
- PLS was introduced in
  David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. "How easy is local search?" In: Journal of computer and system sciences 37.1 (1988), pp. 79–100.
- With associated notions of PLS-reductions and PLS-complete problems.

Approximation Schema

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- For any r, $\mathcal{A}(x, r)$ is an r-approximation algorithm.
- However, for an NP-hard NPO problem, the time performed by the algorithm should increase as r approaches to 1.

PTAS and FPTAS

An optimization problem belongs to

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  This insures, polynomial time algorithms even for values of r that are not constant.
- Usually there is no distinction in the name of the complexity class

A problem in FPTAS: 0-1 Knapsack

0-1 KNAPSACK

Given an integer b and a set of n objects, object i has weight w_i and value v_i, compute a selection of objects with total size less than or equal to b and maximum profit.

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Given an integer b and a set of n objects, object i has weight w_i and value v_i, compute a selection of objects with total size less than or equal to b and maximum profit.

The problem is NP-hard and belongs to NPO. There is a dynamic programming algorithm that solves 0-1 KNAPSACK in time

$$\sim n \sum_{i=1}^{n} v_i.$$

The algorithm is polynomial for $poly(n)$ values.

Consider the following SCALEDOWN algorithm which has r as input:

SCALEDOWN(w, v, b, r)

$v_{\max} := \max v_i$;

$t := \lfloor \log[\frac{r-1}{r} \frac{v_{\max}}{n}] \rfloor$;

$z :=$ instance obtained by changing profits to $v_i' = v_i/2^t$;

$y :=$ optimal solution for z;

return y

Is SCALEDOWN a polynomial time approximation schema? time? rate of approximation?

Time

Time

The most difficult part is the computation of the optimal solution that takes time

$$n\sum_{i=1}^{n} v_i' = n\frac{\sum_{i=1}^{n} v_i}{2^t} = n^2\frac{v_{\max}}{2^t}$$

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But, $t = \lfloor \log \frac{r-1}{r} \frac{v_{\max}}{n} \rfloor$
Thus, for $r \to 1$, $= n^2 \frac{v_{\max}}{\frac{r-1}{r}\frac{v_{\max}}{n}} = \frac{rn^3}{r-1} = O(\frac{n^3}{r-1})$.

polynomial in input size and $1/(r-1)$

Quality of the solution

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$\text{opt}(x) - A(x, r) \leq n2^t$, and $nv_{\max} \geq \text{opt}(x) \geq v_{\max}$.

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$$\frac{n2^t}{v_{\max}} \geq \frac{\text{opt}(x) - A(x,r)}{\text{opt}(x)} = 1 - \frac{A(x,r)}{\text{opt}(x)}$$

$$\frac{A(x,r)}{\text{opt}(x)} \geq 1 - \frac{n2^t}{v_{\max}} = \frac{v_{\max} - n2^t}{v_{\max}}$$

But $t = \lfloor \log \frac{r-1}{r} \frac{v_{\max}}{n} \rfloor$ and $n2^t = n\frac{r-1}{r}\frac{v_{\max}}{n} = v_{\max}\frac{r-1}{r}$.

$$\text{opt}(x) \leq \frac{v_{\max}}{v_{\max} - n2^t} A(x,r) = \frac{v_{\max}}{v_{\max} - \frac{V_{\max}(r-1)}{r}} A(x,r) \leq rA(x,r).$$

Thus, SCALEDOWN is an r-approximation, we have a FPTAS for KNAPSACK.

Min-TSP with triangle inequality

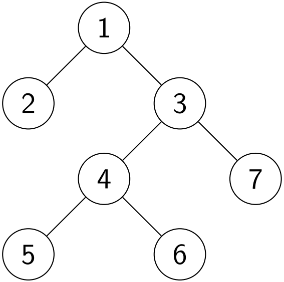
MIN-M-TSP

Given a set of n cities together with distances among any pair of cities, under the assumption that distances verify the triangle inequality, find a shortest tour.

We model the instance by a weighted graph $G = (V, E, d)$.

Algorithm TSP-ST

- Compute a minimum spanning tree T of G.
- Find the directed graph T' obtained from T by replacing each edge with two arcs in oposite directions.
- Find an *Eulerian circuit* R of T'.
- Let S be the walk in G directed by R.
- Transform S in a tour C, by removing (in order) all the vertices that have been already visited in S.



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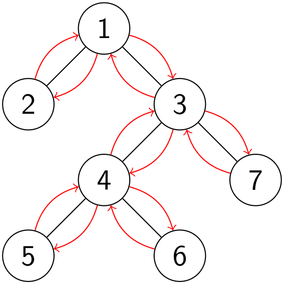

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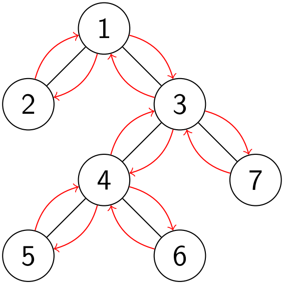

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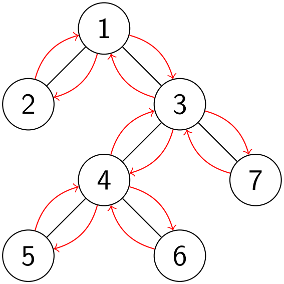
Walk $S = 1213454643731$



MST T Directed T'

Walk $S = 1213454643731$

Tour $S = 12345671$

Theorem

Algorithm TSP-ST *is a polynomial 2-aproximation algorithm for* MIN-M-TSP.

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Proof.

Observe that:

- $\text{opt}(G) \leq c(C) \leq c(S) = c(R)$ due to triangle inequality.
- $c(R) = 2c(T)$ as we use each edge twice.
- Furthermore, any circuit provides a spanning tree, just by removing one of their edges, total cost is below the circuit's distance: $c(T) \leq \text{opt}(G)$.

☺

Christofides' algorithm TSP-CH

Handshaking lemma: every finite undirected graph has an even number of vertices with odd degree

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- Let O be the set of vertices with odd degree in T. By the handshaking lemma, O has an even number of vertices.

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- Compute a minimum spanning tree T of G.
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- Find a minimum-weight perfect matching M in the induced subgraph given by the vertices from O.

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- Combine the edges of M and T to form a connected multigraph H in which each vertex has even degree.

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- Transform S in a tour C, by removing (in order) all the vertices that have been already visited in S.

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TSP is not approximable

MIN-TSP

Given a set of n cities together with weights among any pair of cities, find a shortest tour.

Theorem

MIN-TSP *is non-approximable.*

Proof

Assume that we have a polynomial time $r(n)$-approximation algorithm $\mathcal{A}$, where $r(n)$ requires polynomial number of bits.

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Given a graph G with n vertices consider the instance of Min-TSP on n cities and weights:

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