References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Algorithms for approximation, parameterization, and data streams

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

1 References and basics

2 Approximation algorithms

3 Greedy

4 Local Search

5 Scaling

6 Combinatorial algorithms

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

We will cover

- Optimization/decision/on-line problems
- Approximation and Parameterized complexity
  - Definition
  - Examples
- Some techniques for the design of approximation algorithms
- Linear Programming and approximation
- Some techniques for the design of FPT algorithms
- Some algorithms for data streams

Not necessarily in this order.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

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(S. = Springer)

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

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(S. = Springer)

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

References: Data Streaming

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Problems

Examples:

- Given a graph and two vertices, is there a path joining them?
  decison problem

- Given a graph and two vertices, obtain a path joining them
  with minimum length.
  optimization problem

- Given a set of vertices and two of them, obtain a path joining
  them with minimum length, as time passes, in the graph
  discovered by accessing the sequence of edges $e_1, e_2, \ldots$.
  The algorithm needs to answer the question at any time step
  without knowing the future edges in the graph.
  data stream problem

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Decision Problems: Complexity classes

- P polynomial time
  there is a polynomial time algorithm providing the correct
  answer. for any input.

- EXP exponential time optimization
  there is an exponential time algorithm providing the correct
  answer. for any input.

- NP non-deterministic polynomial time
  Syntactic definition!

  You already known about these classes!

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Optimization Problems

An optimization problem is a structure $\mathcal{P} = (\mathsf{I}, \mathrm{sol}, \mathrm{m}, \mathrm{goal})$, where

- I is the input set to $\mathcal{P}$;
- $\mathrm{sol}(x)$ is the set of feasible solutions for an input x.
- The objective function m is an integer (rational) measure defined over pairs (x, y), for $x \in I$ and $y \in \mathrm{sol}(x)$.
- goal is the optimization criterium MAX or MIN.

That is the function problem whose goal, with respect to an instance x, is to find an optimum solution, that is, a feasible solution y such that

$$y = \mathrm{goal}\{(m(x, y') \mid y' \in \mathrm{sol}(x)\}.$$

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Optimization Problems: Complexity classes

- PO polynomial time optimization
  there is a polynomial time algorithm computing an optimal
  solution for any input.

- EXPO exponential time optimization
  there is an exponential time algorithm computing an optimal
  solution for any input.

- NPO NP optimization
  Syntactic definition (next slide)

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

The NPO class

An optimization problem $\mathcal{P} = (\mathrm{I}, \mathrm{sol}, \mathrm{m}, \mathrm{goal})$ belongs to NPO iff

- I is recognizable in polynomial time.
- The feasible solutions are short:
  a polynomial p exists such that, for $y \in \mathrm{sol}(x)$, $|y| \leq p(|x|)$.
  Moreover, it is decidable in polynomial time whether
  $y \in \mathrm{sol}(x)$, for x, y with $|y| \leq p(|x|)$,
- The objective function m is computable in polynomial time.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

The NPO class: Hardness

- The bounded version of an optimization problem is the decision problem

    - minimization $\mathcal{P} = (I, sol, m, \min)$ is

    Given $x \in I$ and an integer k
    Is there a solution $y \in sol(x)$ such that $m(x, y) \leqslant k$?

    - maximization $\mathcal{P} = (I, sol, m, \max)$ is

    Given $x \in I$ and an integer k
    Is there a solution $y \in sol(x)$ such that $m(x, y) \geqslant k$?

- A NPO problem is NP-hard if its bounded version is NP-complete.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

The NPO class: Justification

- The bounded version of a NPO problem belongs to NP.

  Why?

- For a NPO problem the bounded version and the optimization problem are polynomially equivalent in the following sense:

  There is a polynomial time algorithm for the bounded version iff there is a polynomial time algorithm to compute the cost of an optimal solution

  Why?

- Can an NPO problem be NP-complete?

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Some NPO problems

MIN-BIN PACKING
Given n objects, object i has volume v_i, $0 \le v_i \le 1$, compute the minimum number of unit bins needed to pack all the objects.

MAX-SAT
Given a CNF formula F, compute an assignment that satisfies the maximum number of clauses.

MAX-W-SAT
Given a CNF formula F, in which each clause has an assigned weight. Define the value of an assignment as the sum of the weights of the satisfied clauses. The problem consists in computing an assignment with maximum value.

And the bounded literal per clause families MAX-K-SAT.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

Some NPO problems: hardness

Which problems in the previous slide are NP-hard? Why?

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Problem types
Complexity classes
Algorithmic techniques for hard problems

How to "solve" NP-hard optimization problems in practice

We know that an optimization problem, whose decision version is NP-hard, cannot be solved in polynomial time unless P $=$NP.
What to do?

- Exact algorithms: for instances of small size or for restricted classes of instances.
- Randomization: algorithms that use random bits, either with high probability of succes or with high probability of poly time.
- Heuristic method: a feasible solution with empirical guarantee.
- Approximation algorithm: a feasible solution with a performance guarantee.
- Parameterization: solve efficiently some slices of the problem.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Definition
Complexity classes

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Definition
Complexity classes

Approximation algorithms

- Let $\mathcal{P}$ be an optimization problem.
- For any instance x of $\mathcal{P}$ let opt(x) be the cost of an optimal solution.
- Let $\mathcal{A}$ be an algorithm such that for any instance x of $\mathcal{P}$ computes a feasible solution with cost $\mathcal{A}(x)$.

$\mathcal{A}$ is an *r-approximation* for $\mathcal{P}$ ($r \geq 1$) if for any instance x of $\mathcal{P}$

$$\frac{1}{r} \leq \frac{\text{opt}(x)}{\mathcal{A}(x)} \leq r$$

$\mathcal{P}$ is *r-approximable in polynomial time* if there is a polynomial time computable *r*-approximation for $\mathcal{P}$.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Definition
Complexity classes

Be sure about r

$\mathcal{A}$ is an *r-approximation* for $\mathcal{P}$ ($r \geq 1$) if for any instance x of $\mathcal{P}$

$$\frac{1}{r} \leq \frac{\text{opt}(x)}{\mathcal{A}(x)} \leq r$$

- Why $r \geq 1$?
- In which cases we can have $r = 1$?
- How would you like r to be?

- Is there any trivial condition about r, for maximization problems? for minimization ones?

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Definition
Complexity classes

NPO: approximation classes

Classification of NPO problems as approximable within a constant r in polynomial time:

APX exists r.

PTAS Polynomial Time Approximation Schema
for any r, but time may depend exponentially in $1/(r-1)$.

FPTAS Fully Polynomial Time Approximation Schema
for any r in time polynomial in the input size and $1/(r-1)$.

Further classification can be obtained by considering non-constant r, for example $\log n$ or $\log \log n \ldots$.

Negative results through hardness APX-hard etc.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Definition
Complexity classes

Hard to approximate problems

Hardness levels:

APX-hard: unless P $=$NP no PTAS
unless P $=$NP no constant approximation
⋮
Non-approximable
unless P $=$NP, for any r at most a polynomial function of n, there
is no polynomial time r-approximation algorithm

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

1 References and basics

2 Approximation algorithms

3 Greedy

4 Local Search

5 Scaling

6 Combinatorial algorithms

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

First approximation algorithms

- We start analyzing approximation algorithms based in the greedy technique.
- One algorithm for MIN-BIN PACKING
- And two algorithms for a load-balancing problem.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Nex Fit: An approximation algorithm for BinPacking

MIN-BIN PACKING

Given n objects, object i has volume v_i, $0 \leq v_i \leq 1$, compute the minimum number of unit bins needed to pack all the objects.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Nex Fit: An approximation algorithm for BinPacking

Let us assume that the bins are labeled $B_1, B_2, \ldots, B_n$ and that initially all are closed and during the execution of the algorithm only one bin will be open.

Next Fit places the n objects, one after the other, as follows:

- Opens B_1 and places the first object in B_1.
- If the i-th object fits in the open box, we put it inside. Otherwise, we close the bin, open the next one and place the object in it.

For the case of $v_1 = 0.3$, $v_2 = 0.8$, and $v_3 = 0.7$, Next Fit solution needs three bins. But there is a solution with uses only two bins. Not optimal!

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Nex Fit: An approximation algorithm for BinPacking

Theorem

Let x be an input to the MIN-BIN PACKING *problem and let $opt(x)$ be the minimum number of bins needed to pack de objects in x. If $NF(x)$ is the number of bins in the solution computed by Next Fit, then $opt(x) \leq NF(x) \leq 2opt(x)$.*

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Nex Fit: An approximation algorithm for BinPacking

Theorem

Let $x = (v_1, \ldots, v_n$ be an input to the MIN-BIN PACKING. Let
$NF(x)$ be the number of bins in the solution computed by Next
Fit, we have $opt(x) \leq NF(x) \leq 2opt(x)$.

Proof.

The first inequality is always true:

- MIN-BIN PACKING is a minimization problem
- Next Fit provides a feasible solution

Let $V = \sum_{i=1}^{n} v_i$, we have $opt(x) \geq \lceil V \rceil$.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Nex Fit: An approximation algorithm for BinPacking

Proof.

Let us look to two consecutive bins in the Next Fit solution The total packet size in the two bins must be bigger than 1, otherwise we will never have opened the second bin. So,

$$NF(x) \leq 2\lceil V \rceil.$$

But we have seen $opt(x) \geq \lceil V \rceil$, so

$$NF(x) \leq 2\lceil V \rceil \leq 2opt(x).$$

☺

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Load Balancing problem

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Processing scenario

- We have m identical machines;
  n jobs, job j has processing time t_j
- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.
- We want to assign jobs to machines optimizing the makespan.

- Let $J(i)$ be the subset of jobs assigned to machine i.
- The load of machine i is $L_i = \sum_{j \in J(i)} t_j$
- The makespan is the maximum load on any machine,
  $L = \max_i L_i$.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Load Balancing problem

LBAL

Given n jobs, job j has processing time t_j, assign jobs to m identical machines as to minimize the makespan.

- The problem is NP-hard and belongs to NPO.
  Load balancing is hard even if $m = 2$ machines
  (reduction from Partition).

- The approximation algorithm we propose is a greedy algorithm called list-scheduling.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

List scheduling

LIST SCHEDULING

For $j = 1, \ldots, n$:

Assign job j to the machine having smallest load so far.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

List scheduling: Implementation

function LIST SCHEDULING(m, n, T)
    for $i = 1, dots, m$ **do**
        $L[i] = 0$ load on machine
        $J[i] = \emptyset$ jobs assigned to
    end for
    for $j = 1, \ldots, n$ **do**
        $i = \text{argmin}_k L_k$ machine with smallest load
        $J[i] = J[i] \cup \{j\}$
        $L[i] = L[i] + T[j]$
    end for
end function

Cost: Using a priority queue to maintain L, the cost is $O(n \log m)$

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

List scheduling: Approximation rate

Theorem

LIST SCHEDULING *is a polynomial 2-approximation algorithm for* LBAL.

Proof.

- Let L^* be the optimum makespan and L the makespan of the solution computed by list scheduling.
- $L^* \geq \max_j t_j$ and $L^* \geq \frac{1}{m} \sum_j t_j$.
- Assume that $L = L_i$. Let j be the last job scheduled in machine i.
- When job j was assigned, all the other machines have higher load, so $L_i - t_j \leq L_k$, for $k \neq i$.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

List scheduling: Approximation rate

Theorem

LIST SCHEDULING *is a polynomial 2-approximation algorithm for* LBAL.

Proof.

- When job j was assigned, all the other machines have higher load, so $L_i - t_j \leq L_k$, for $1 \leq k \leq m$. Summing up for all k and dividing by m we get
- $L_i - t_j \leq \frac{1}{m} \sum_k L_k \leq \frac{1}{m} \sum_j t_j \leq L^*$
- We have, $L = L_i = (L_i - t_j) + t_j \leq L^* + L^* = 2L^*$

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Approximation rate: tightness

- When we design an approximation algorithm, we wish to approach the best possible approximation ratio.

- Which one is the best for a problem, as usual, requires some complexity consideration and, in some cases, we have answers like
  "this problem cannot be approximated for $r \leq \ldots$
  unless P $=$NP"

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Approximation algorithm: tightness

- We can ask a similar tightness question, not for the optimization problem, but about the approximation algorithm at hand.

- In this case the question is about the tightness in the analysis of the approximation ratio. The value of r is correct or can it be reduced further?

- We can show the tightness in the analysis of r by finding an input x so that the rate between $\mathrm{opt}(x)$ and $\mathcal{A}(x)$ rules out any improvement on r.

- The previous step involves computing the optimal solution for a particularly adequate input, not solving the optimization problem.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

List scheduling: Tightness?

- m machines, $m(m-1)$ length 1 jobs and 1 job of length m.
- List scheduling assigns the first m length 1 jobs, to different machines, and
- after scheduling the unit length jobs, all machines have load $L_i = m - 1$.
- The last job is assigned to one machine, giving a makespan $L = m + m - 1 = 2m - 1$.
- An optimal solution assigns the big job to one machine and m unit jobs to the other machines, so $L^* = m$.
- The approximation rate is tight ☺

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Longest processing first

- Analizamos una variante,

  LONGEST PROCESSING FIRST:
    Sort jobs in decreasing order of processing time.
    Run list scheduling.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Longest processing first: Approximation rate

Theorem

LONGEST PROCESSING FIRST *is a polynomial 3/2-approximation algorithm for* LBAL.

Proof.

- If $n \leq m$, $L = t_1 = L^* \leq 3/2L^*$
- If $n > m$, since there are more jobs than machines, a machine must take two jobs in $(t_1, \ldots, t_{m+1})$.
- Therefore, $L^* \geq 2t_{m+1} \geq 2t_j$, being j the last job assigned to the machine i that gives the makespan.
- So, $L_i = (L_i - t_j) + t_j \leq L^* + \frac{1}{2}L^* \leq \frac{3}{2}L^*$.

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Bin Packing
Load Balancing

Longest processing first: Tightness?

- The $3/2$ bound on the approximation rate is not tight
- In fact LONGEST PROCESSING FIRST is a $4/3$-approximation algorithm [Graham 1969]
- $4/3$ is tight:
    - m machines
    - $n = 2m + 1$ jobs
    - 2 jobs of length $m, m + 1, \ldots, 2m - 1$ and one more job of lenght m.
    - $L^* = 3m$ and $L = 4m - 1$, which gives a ratio tending to $4/3$.
    - $4/3$ is tight ☺

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

Max Cut

MAX-CUT

Given a graph $G = (V, A)$ we want to find a partition of V into V_1, V_2 in such a way that

$$\text{cut}(V_1, V_2) = \|\{(u, v) \mid u \in V_1 \ v \in V_2\}\|$$

is maximum.

- The problem is NP-hard and belongs to NPO.
- Let us analyze a local search algorithm using the HillClimbing paradigm.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

Local search

- A neighborhood structure is defined on the set of solutions.
- The algorithm performs an exploration of the neighborhood graph.
- Hill Climbing: It starts at one feasible solution and moves to a better one. It finishes at a local optimum, when no neighbor improves the value of m.
- Many heuristics are local search algorithms performing some kind of random exploration on the neighborhood. Te result of such an exploration is the best seen solution.

A local optimum is a solution such that all its neighbors have equal or worse cost.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

Neighborhood for MaxCut

Given a graph $G = (V, E)$ define $\mathcal{N}(G)$:

> *The neighbors of a solution (V_1, V_2) are all those partitions that can be obtained by moving either one element from V_1 to V_2 or one element from V_2 to V_1.*

Using $\mathcal{N}(G)$ we consider the following algorithm:

> *HillClimbing Max-Cut (G:graph, n: integer)*
> $\quad V_1, V_2$: *set of* $[1 \ldots n]$;
> $\quad V_1 := \varnothing$; $V_2 := \mathsf{V}(G)$;
> $\quad$ *while not local-optimum(V_1, V_2) do*
> $\quad\quad (V_1, V2) := $ *a neighboring partition of* (V_1, V_2)
> $\quad\quad$ *with improved cost*

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

HillClimbing MaxCut

Observe that the algorithm needs only polynomial time as

- we can compute $\text{cut}(X, Y)$ in $O(|E(G)|)$ steps.
  In fact you can recompute faster from the value of a neighbor.
- The number of neighbors of a solution is n.
- The cost of the initial solution is 0.
- The maximum partition cut is upper bounded by $|E(G)|$.
- At each step the cut is increased in one unit.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

HillClimbing MaxCut

Lemma

Let $G = (V, A)$ be a graph, if (V_1, V_2) is a local optimum of $\mathcal{N}(G)$ then $cut((V_1, V_2)) \leq 2opt(G)$.

Theorem

HILLCLIMBING MAX-CUT is a polynomial 2-approximation algorithm for MAX-CUT.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

The class PLS

- Polynomial Local Search (PLS) is a complexity class that models the difficulty of finding a locally optimal solution to an optimization problem.
- A Local search problem (*LSP*) is an optimization problem together with a neighborhood defined on the set of solutions.
- A *LSP* problem $\mathcal{P} = (\mathrm{I}, \mathrm{sol}, \mathrm{m}, \mathrm{goal}, \mathcal{N})$ belongs to PLS if
  - $(\mathrm{I}, \mathrm{sol}, \mathrm{m}, \mathrm{goal}) \in \mathrm{NPO}$.
  - Given $x \in \mathrm{I}$, a $y \in \mathrm{sol}$ can be computed in polynomial time.
  - Given $y \in \mathrm{sol}(x)$, there is a polynomial (in $|x|$) algorithm that decides whether y is a local optimum, for m and, if not, outputs a neighbor of y with better cost.
- PLS problems always have a solution!

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

MaxCut

The class PLS

- The conditions guarantee that a navigation step can be performed in polynomial time.
- The size of the solution set do not guarantee that an exploration will end within polynomial time.
- However the computation uses only polynomial space.
- PLS was introduced in
  David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. "How easy is local search?" In: Journal of computer and system sciences 37.1 (1988), pp. 79–100.
- With associated notions of PLS-reductions and PLS-complete problems.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Approximation Schema

- An approximation scheme is an algorithm $\mathcal{A}$ that takes as input an instance of an optimization problem and a parameter $r \geq 1$ and outputs a solution with cost within r of the optimal solution.

- For any r, $\mathcal{A}(x, r)$ is an r-approximation algorithm.

- However, for an NP-hard NPO problem, the time performed by the algorithm should increase as r approaches to 1.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

PTAS and FPTAS

An optimization problem belongs to

- Polynomial Time Approximation Scheme (PTAS) if it has an approximation scheme $\mathcal{A}$ that takes time polynomial in $|x|$ independently of the dependency on $\frac{1}{r-1}$.
  This insures, polynomial time for any constant r.

- Fully Polynomial Time Approximation Scheme (FPTAS) if it has an approximation scheme $\mathcal{A}$ that takes time polynomial in both $p(|x|)$ and $\frac{1}{r-1}$.
  This insures, polynomial time algorithms even for values of r that are not constant.

- Usually there is no distinction in the name of the complexity class

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

A problem in FPTAS: 0-1 Knapsack

0-1 KNAPSACK

Given an integer b and a set of n objects, object i has weight w_i and value v_i, compute a selection of objects with total size less than or equal to b and maximum profit.

The problem is NP-hard and belongs to NPO. There is a dynamic programming algorithm that solves 0-1 KNAPSACK in time

$$\sim n \sum_{i=1}^{n} v_i.$$

The algorithm is polynomial for $poly(n)$ values.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Consider the following SCALEDOWN algorithm which has r as input:

> SCALEDOWN(w, v, b, r)
>
> $v_{\max} := \max v_i$;
>
> $t := \lfloor \log[\frac{r-1}{r} \frac{v_{\max}}{n}] \rfloor$;
>
> $z :=$ instance obtained by changing profits to $v_i' = v_i/2^t$;
>
> $y :=$ optimal solution for z;
>
> **return** y

Is SCALEDOWN a polynomial time approximation schema? time? rate of approximation?

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Time

The most difficult part is the computation of the optimal solution that takes time

$$n \sum_{i=1}^{n} v_i' = n \frac{\sum_{i=1}^{n} v_i}{2^t} = n^2 \frac{v_{\max}}{2^t}$$

But, $t = \lfloor \log \frac{r-1}{r} \frac{v_{\max}}{n} \rfloor$

Thus, for $r \to 1$, $= n^2 \frac{v_{\max}}{\frac{r-1}{r} \frac{v_{\max}}{n}} = \frac{rn^3}{r-1} = O(\frac{n^3}{r-1})$.

polynomial in input size and $1/(r-1)$

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Quality of the solution

$\mathrm{opt}(x) - A(x, r) \leq n2^t$, and $nv_{\max} \geq \mathrm{opt}(x) \geq v_{\max}$. Thus

$$\frac{n2^t}{v_{\max}} \geq \frac{\mathrm{opt}(x) - A(x, r)}{\mathrm{opt}(x)} = 1 - \frac{A(x, r)}{\mathrm{opt}(x)}$$

$$\frac{A(x, r)}{\mathrm{opt}(x)} \geq 1 - \frac{n2^t}{v_{\max}} = \frac{v_{\max} - n2^t}{v_{\max}}$$

But $t = \lfloor \log \frac{r-1}{r} \frac{v_{\max}}{n} \rfloor$ and $n2^t = n\frac{r-1}{r}\frac{v_{\max}}{n} = v_{\max}\frac{r-1}{r}$.

$$\mathrm{opt}(x) \leq \frac{v_{\max}}{v_{\max} - n2^t}A(x, r) = \frac{v_{\max}}{v_{\max} - \frac{v_{\max}(r-1)}{r}}A(x, r) \leq rA(x, r).$$

Thus, SCALEDOWN is an r-approximation, we have a FPTAS for KNAPSACK.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

1 References and basics

2 Approximation algorithms

3 Greedy

4 Local Search

5 Scaling

6 Combinatorial algorithms

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

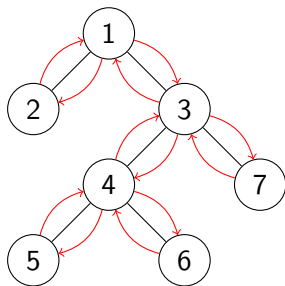## Min-TSP with triangle inequality

MIN-M-TSP
Given a set of n cities together with distances among any pair of cities, under the assumption that distances verify the triangle inequality, find a shortest tour.

We model the instance by a weighted graph $G = (V, E, d)$.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Algorithm TSP-ST

- Compute a minimum spanning tree T of G.
- Find the directed graph T' obtained from T by replacing each edge with two arcs in oposite directions.
- Find an *Eulerian circuit* R of T'.
- Let S be the walk in G directed by R.
- Transform S in a tour C, by removing (in order) all the vertices that have been already visited in S.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

MST T Directed T'

Walk $S = 1213454643731$

Tour $S = 12345671$

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Theorem

Algorithm TSP-ST *is a polynomial 2-aproximation algorithm for* MIN-M-TSP.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Theorem

Algorithm TSP-ST *is a polynomial 2-aproximation algorithm for* MIN-M-TSP.

Proof.

Observe that:

- $\text{opt}(G) \leq c(C) \leq c(S) = c(R)$ due to triangle inequality.
- $c(R) = 2c(T)$ as we use each edge twice.
- Furthermore, any circuit provides a spanning tree, just by removing one of their edges, total cost is below the circuit's distance: $c(T) \leq \text{opt}(G)$.

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Christofides' algorithm TSP-CH

Handshaking lemma: every finite undirected graph has an even number of vertices with odd degree

- Compute a minimum spanning tree T of G.
- Let O be the set of vertices with odd degree in T. By the handshaking lemma, O has an even number of vertices.
- Find a minimum-weight perfect matching M in the induced subgraph given by the vertices from O.
- Combine the edges of M and T to form a connected multigraph H in which each vertex has even degree.
- Find an *Eulerian circuit* R of H.
- Let S be the walk in G directed by R.
- Transform S in a tour C, by removing (in order) all the vertices that have been already visited in S.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Lemma

$G = (V, E)$ is a graph. Let M be a minimum-weight perfect matching for O. Then $c(M) \leq opt(G)/2$.

Proof.

- Take any optimal tour of G, take the shortcuts to make a tour K for O. By the triangle inequality $c(K) \leq \text{opt}(G)$

- As O has even number of vertices K can be decomposed into two (alternating) perfect matchings S and S'.
  M is a minimum weight perfect matching and
  $2c(M) \leq c(S) + c(S') = c(K) \leq \text{opt}(G)$

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

Theorem

Algorithm TSP-CH *is a polynomial 3/2-aproximation algorithm for* MIN-M-TSP.

Proof.

- By the previous lemma $c(M) \leq \text{opt}(G)/2$.
- We already know that $c(T) \leq \text{opt}(G)$
- Also, $\text{opt}(G) \leq c(C) \leq c(S) = c(R)$ due to triangle inequality.
- By construction, $c(R) \leq c(T) + w(M) \leq \text{opt}(G) + \frac{\text{opt}(G)}{2}$

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References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

TSP is not approximable

MIN-TSP

Given a set of n cities together with weights among any pair of cities, find a shortest tour.

Theorem

MIN-TSP *is non-approximable.*

Proof

Assume that we have a polynomial time $r(n)$-approximation algorithm $\mathcal{A}$, where $r(n)$ requires polynomial number of bits.

References and basics
Approximation algorithms
Greedy
Local Search
Scaling
Combinatorial algorithms

Min-TSP with triangle inequality

TSP is not approximable

Given a graph G with n vertices consider the instance of Min-TSP on n cities and weights:

$$w(i,j) = \begin{cases} 0 & \text{if } (i,j) \in E \\ r(n)n & \text{otherwise} \end{cases}$$

- If G has a Hamiltonian Circuit, there is a TSP circuit with weight n, so $\mathcal{A}(G) \leq r(n)n$.
- If G has no Hamiltonian Circuit, any TSP circuit has weight $> nr(n)$, so $\mathcal{A}(G) > r(n)n$.

But, as $\mathcal{A}$ is polynomial, P=NP!

End Proof