Algorithms for data streams: Streaming models, a problem on graph streams

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- We abstract the data to a particular feature, the data field of interest the label.

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- Take into account that some times we do not know in advance the length of the stream.
- Goal Compute a function of stream, e.g., median, number of distinct elements, longest increasing sequence.

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- Theoretical Appeal:
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- Origins in 70's but has become popular in this century because of growing theory and very applicable.

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- Streaming with sorting
  - Allows the creation of intermediate streams.
  - Streams can be sorted at no cost.
  - Algorithms run in phases reading and creating a stream

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- Algorithms use randomization and seek for an approximate answer.
- Typical approach:
  - Build up a synopsis data structure
  - It should be enough to compute answers with a high confidence level.

Streams that describe graphs

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- on $[n]$ vertices
- the stream describes the edges of G
- we assume that an edge appears only once in the stream
- We want to keep a DS allowing to answer queries about a graph property
- $O(n \log n)$ memory is reasonable as we are working on the semi-streaming model.

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- Algorithm:

1: **procedure** EDISTANCES(int n, stream s, double t)
2: $H = ([n], \emptyset)$
3: **while** not $s.end()$ **do**
4: $(u, v) = s.read()$
5: **if** $d_H(u, v) \geq t + 1$ **then**
6: add (u, v) to H
7: update distances
8: On query (u, v), report $d_H(u, v)$

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- $|H|$?
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- A subgraph H is a t-spanner for G iff, for all $u, v \in V$ $(V = V(G) = V(H))$,

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  - Pick $i \in [k]$ and let $e = (v_{i-1}, v_i)$.
    If $e \in H$, $d_H(v_{i-1}, v_i)) = 1$, otherwise, as e was not added to H, we known that in the current H' when e was read $d_{H'}(v_{i-1}, v_i) \leq t$.

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  - So, $d_H(x, y) \leq t\, d_G(x, y)$ as any such H' is a subgraph of H.

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  - So, $d_H(x, y) \leq t\, d_G(x, y)$ as any such H' is a subgraph of H.
- Then, EDISTANCES is a t-approximation

Estimating shortest paths: Analysis

The girth $\gamma(G)$ is the length of the shortest cycle in G, ∞ for acyclic G.

Alon, Hoory, Linial 2002

Let n be sufficiently large. Let G be a graph with n vertices and m edges having $\gamma(G) \geq k$, for some integer k. Then

$$m \leq n + n^{1+\lfloor \frac{k-1}{2} \rfloor}$$

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- We can 3-approximate distances using $O(n^{5/3} \log n)$ memory.

Maximum matching

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```
1: procedure MMatching(int n, stream s, double t)
2:     M = ∅
3:     while not s.end() do
4:         (u, v) = s.read()
5:         if M ∪ (u, v) is a matching then
6:             M = M ∪ {(u, v)}
7:     On query, report M
```

Maximum matching: Analysis

Maximum matching: Analysis

- 1 pass, $O(n \log n)$ space, $O(1)$ ops. per item
- M is a maximal matching that provides an estimation $\hat{f}$ of the size f of a maximum matching.
- $\hat{f}$ is a 2 approximation to f.

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- Example: To compute the median packet size of some IP packets, we could just sample some and use the median of the sample as an estimate for the true median. Statistical arguments relate the size of the sample to the accuracy of the estimate.
- Challenge: But how do you take a sample from a stream of unknown length or from a sliding window?

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  - Quality?
    What is the probability that $x = x_i$ at some time $t \geq i$?

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  - Base $t = 1$: $Pr[x = x_1] = 1$.
  - Induction hypothesis: true for time steps up to $t - 1$
    - $Pr[x = x_t] = 1/t$
    - For $i < t$, $x = x_i$ only when x_t is not selected and x_i was the sampled element at step $t - 1$. By induction hypothesis we have

$$Pr[x = x_t] = \left(1 - \frac{1}{t}\right) \frac{1}{t-1} = \frac{1}{t}$$

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  - Initially $X = \{x_1, \ldots, x_k\}$.
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    - $Pr[x_t \in X] = k/t$
    - For $i < t$, $x_i \in X$ when x_t is not selected and x_i was in the sample at step $t - 1$, or when x_t is selected, x_i was in the sample at step $t - 1$ and x_i is not evicted.

$$Pr[x_i \in X] = \left(1 - \frac{k}{t}\right)\frac{k}{t-1} + \frac{k}{t}\frac{k}{t-1}\left(1 - \frac{1}{k}\right)$$
$$= \frac{k}{t-1} - \frac{k}{t}\frac{k}{t-1}\frac{1}{k} = \frac{k}{t-1} - \frac{1}{t}\frac{k}{t-1} = \frac{k}{t}$$

Sliding Windows: Replace-Sampling algorithm

- Problem: Maintain a uniform sample x from the last seen w elements.
- Algorithm:
  - Maintain a reservoir sample for the first w items in s.
  - When the arrival of an item causes an element in the sample to expire, replace it with the new arrival.

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- Trouble: We keep an element but it might not follow the uniform distribution.

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  - But, we have no access to past data!
  - Could store the entire window but this would require $O(w)$ memory.

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- In probability there are collections of inequalities providing bounds of the probability of being close to the mean under different hypothesis.
- We survey some such useful bounds.

Markov's, Chebyshev's and Chernoff's bounds

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Chebyshev's inequality

Let X be a random variable with finite expected value μ and finite non-zero variance σ^2. Then for any real number $t > 0$,

$$Pr(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}.$$

Chernoff bounds

When applying Markov's inequality to e^{cX}, we get

Chernoff bound

For $c > 0$,

$$Pr(X > a) = Pr(e^{cX} \geq e^{ca}) \leq \frac{E[e^{cX}]}{e^{ca}}$$

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Let $X_1, \ldots, X_n$ be independent random variables corresponding to Bernoulli experiments, each variable with probability p_i of success. Let $X = X_1 + \cdots X_n$ and let $\mu = E[X] = \sum_{i=1}^{n} p_i$. Then, for any $d > 0$,

$$\Pr[X > (1+d)\mu] \leq \left(\frac{e^d}{(1+d)^{1+d}} \right)^{\mu}.$$

For any $0 < d < 1$,

$$\Pr[X < (1-d)\mu] \leq \left(\frac{e^{-d}}{(1-d)^{1-d}} \right)^{\mu}.$$

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$$\Pr[X < (1-d)\mu] \leq e^{\frac{-d^2\mu}{2}} \text{ and } \Pr[X > (1+d)\mu] \leq e^{\frac{-d^2\mu}{3}}.$$

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- The bound on the space is $O(k \log w)$ with high probability.

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  With high probability, for large enough w, such chains will not intersect.

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- With high probability the number of updates is $O(\log m)$.