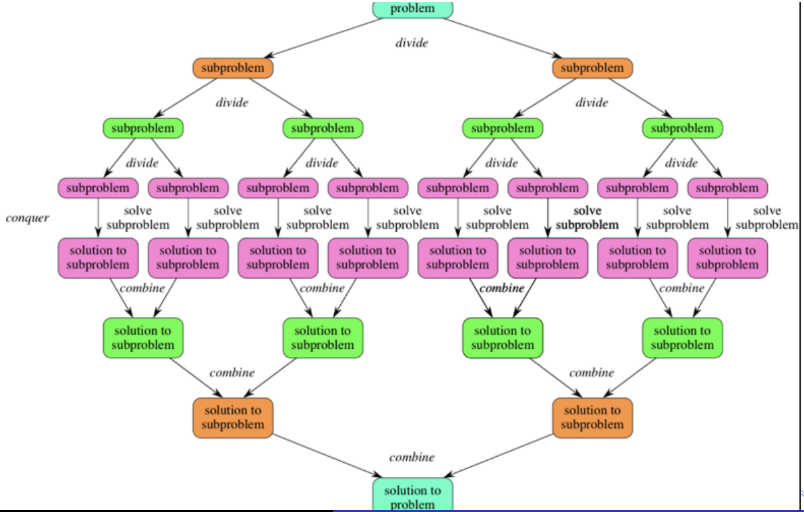
Divide-and-conquer: Selection



The divide-and-conquer strategy.

1. Break the problem into smaller subproblems,
2. recursively solve each problem,
3. appropriately combine their answers.

Known Examples:

- Binary search
- Merge-sort
- Quicksort
- Strassen matrix multiplication

Julius Caesar (I-BC)
"Divide et impera"

J. von Neumann
(1903-57) Merge sort

A basic Master Theorem

There are several versions of the Master Theorem to solve D&C recurrences. The one presented below is taken from DPV's book.

Theorem (DPV-2.2)

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ *for constants* $a \geq 1, b > 1, d \geq 0,$ *then has asymptotic solution:*

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a, \\ O(n^d \lg n), & \text{if } d = \log_b a, \\ O(n^{\log_b a}), & \text{if } d < \log_b a. \end{cases}$$

Master Theorems

- This basic Master Theorem does not provide always exact bounds.
- A different one can be found in CLRS's book providing exact bounds but leaving cases outside.
- For stronger versions look at Akra-Bazi Theorem or Salvador Roura Theorems

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- Selection Problem: Given an array A of n unordered distinct keys, and $i \in \{1, \dots, n\}$, output the rank i element in A, that is the key that is larger than exactly $i - 1$ other keys in A.
- Notice that i can be any rank value between 1 and n, in particular when:
  1. $i = 1$, the MINIMUM element
  2. $i = n$, the MAXIMUM element
  3. $i = \lfloor \frac{n+1}{2} \rfloor$, the MEDIAN

A first algorithm

Sort A in $(O(n \lg n))$ steps, then the i-th smallest key is $A[i]$.

Can we do it faster? in linear time?

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Can we do it faster? in linear time?

Yes, selection is easier than sorting

The algorithm: High level

- Chose a split element x.
- Let k be the rank of x, if $k = i$, we found the i-th element. Otherwise,
- Use x to determine a partition of A, smaller than x to the left and larger to the right.
- Compute recursively the i-th element in the left part, when $i < k$, or the $i - k$-th element in the right part, when $i > k$.

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The time depends on the quality of the splitting element to divide fairly the elements

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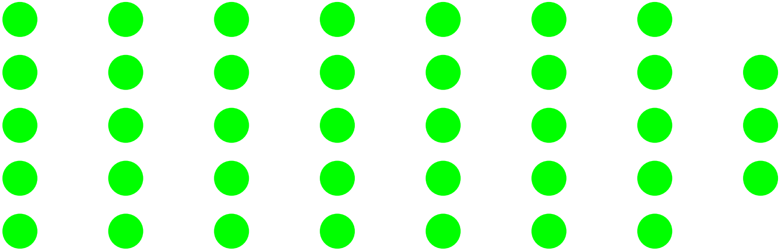

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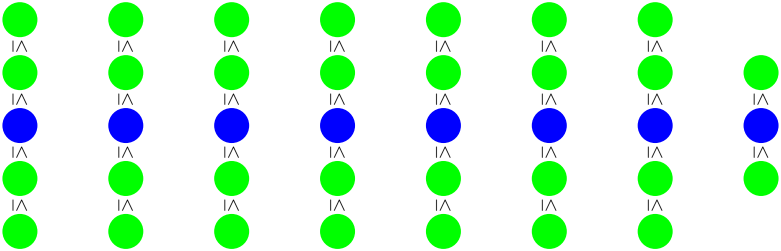

The splitting element x is the median of the set of medians, $\{x_j \mid 1 \leq j \leq \lceil n/5 \rceil\}$.

The algorithm for $n > 5$

Select(A, i)
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements
$X[j]$ = median of group j, $j = 1, \ldots, m$
$x =$ **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X
Let k be the rank of x in A
if $i = k$ **then**
  return x
else
  $L =$ the elements of A smaller than x (left)
  $R =$ the elements of A bigger than x (right)
  if $i < k$ **then**
    return **Select**(L, i)
  else
    return **Select**$(R, i - k)$

Example: Find the median

Let $n = 15$, we want to get the 5-th element on the following input:

$$A = \quad 3\ 13\ 9\ 4\ 5 \quad 1\ 15\ 12\ 10\ 2 \quad 6\ 14\ 8\ 17\ 11$$

An example

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5	10	11
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5	10	11
9	12	14
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The median of $X = (5, 10, 11)$ is 10 which has rank 9
As $5 < 9$, recursively ask for the 5-th element in the left part with respect to $x = 10$, i.e., $(3, 9, 4, 5, 1, 2, 6, 8)$

Example: Find the median

In the next call $n = 8$, we look for the 5-th element in the following input:

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$$\begin{array}{|c|c|}\hline 1 & \\ 3 & 2 \\ \color{red}{4} & \color{red}{6} \\ 5 & 8 \\ 9 & \\ \hline\end{array}$$

The median of $X = (4, 6)$ is $\color{red}{4}$ which has rank 4.

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1	
3	2
4	6
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9	

The median of $X = (4, 6)$ is 4 which has rank 4.
As $5 > 4$ the algorithm looks for the 1st element in the right part $(5, 6, 8, 9)$, which is 5.

Selection algorithm: Cost

Select(A, i)
if $n \leq 5$ **then**
  Sort A and return the i-th element $O(1)$
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements $O(n)$
$X[j]$ = median of group j, $j = 1, \ldots, m$ $O(n)$
x = **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X $T(n/5)$
Let k be the rank of x in A
if $i = k$ **then**
  return x
else
  L = the elements of A smaller than x $O(n)$
  R = the elements of A bigger than x $O(n)$
  if $i < k$ **then**
    return **Select**(L, i) $T(?)$
  else
    return **Select**$(R, i-k)$ $T(?)$

Selection algorithm: elements smaller than x

At least $3\left(\left\lceil \frac{1}{2}\lceil n/5 \rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6$ of the elements are $< x$.

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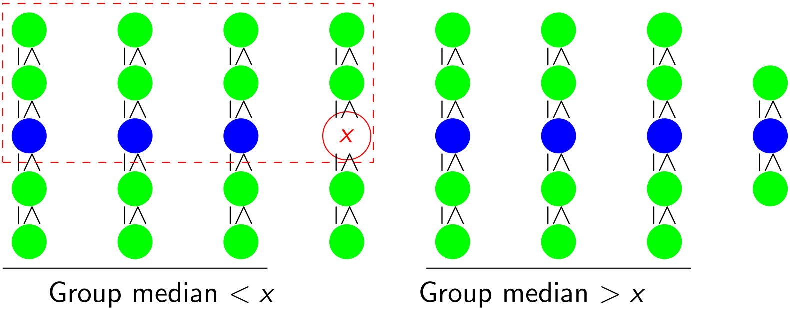

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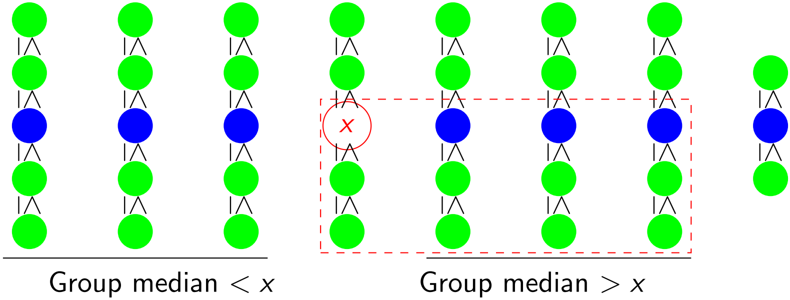

Selection algorithm: the recurrence

- As at least $\geq \frac{3n}{10} - 6$ of the elements are $< x$ $(> x)$, at most $n - (\frac{3n}{10} - 6) = 6 + 7n/10$ elements are $\leq x$ $(\geq x)$.

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Selecting 140 (instead of 5) as the size to stop the recursion, we have

$$T(n) = \begin{cases} \Theta(1) & \text{if } n < 140, \\ T(\lceil n/5 \rceil) + T(6 + 7n/10) + \Theta(n) & \text{if } n \geq 140. \end{cases}$$

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- Let c_1 be a constant, so that $T(n) \leq c_1 n$, for $n \leq 140$.
- Attempt to prove that $T(n) \leq cn$ by induction ($n \geq 140$), and find additional restrictions on c.
- We replace the $\Theta(n)$ term by its exact value dn, for some $d > 0$.

$$
\begin{aligned}
T(n) &\leq T(\lceil n/5 \rceil) + T(6 + 7n/10) + dn \\
&\leq c\lceil n/5 \rceil + c(6 + 7n/10) + dn \\
&\leq c(n/5 + 1) + c(6 + 7n/10) + dn \\
&= 9\frac{cn}{10} + 7c + dn \\
&= cn + (-\frac{cn}{10} + 7c + dn)
\end{aligned}
$$

Which is at most cn, if $-\frac{cn}{10} + 7c + dn \leq 0$.

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- As we assume that $n \geq 140$, choosing $c \geq 20d$ will satisfy the inequality.
- By taking c so that $c \geq c_1$ and $c \geq 20d$, we have that the base case and the inductive step hold, so $T(n) = \Theta(n)$.

Remarks on the size of the groups

Notice:

- If we make groups of 7, the number of elements $\geq x$ is $\frac{2n}{7}$, which yield $T(n) \leq T(n/7) + T(5n/7) + \Theta(n)$ with solution $T(n) = \Theta(n)$.
- However, if we make groups of 3, then $T(n) \leq T(n/3) + T(2n/3) + \Theta(n)$, which has a solution $T(n) = \Theta(n \ln n)$.