Algorithmics: Basic definitions and concepts

The Algorithmics course

Already known (EDA level)

- Algorithms cost and Asymptotic notation

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- Backtracking algorithms

The Algorithmics course

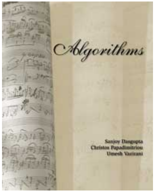
Topics to cover:

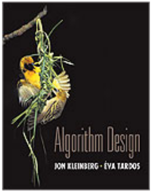
- Divide and conquer: Linear Selection
- Sorting in linear time (when? how?)
- Greedy algorithms
- Dynamic programming
- Distances in graphs
- Flow networks: problems, algorithms and applications
- Linear Programming
- Approximation algorithms
- Streaming algorithms

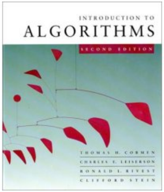
Provide models to solve real problems

References

Main references:







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- Algorithms themselves have evolved into a complex set of techniques, self-learning, Web services, concurrent, distributed or parallel, etc... Each of them with ad-hoc relevant computational limitations and social implications.
- However, this course will be a course on classical algorithms, which are the core needed to understand more advanced computational material.

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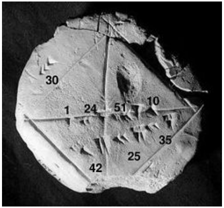
for $i = 1$ to 6 **do**
  $x_i = (x_{i-1} + n/x_{i-1})/2$
end for



Babilònia (XVI BC)

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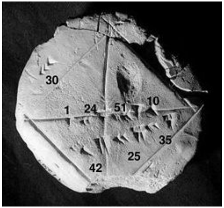

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For $n = 20$, x's are 1 10.5 6.2023 4.7134 4.4783

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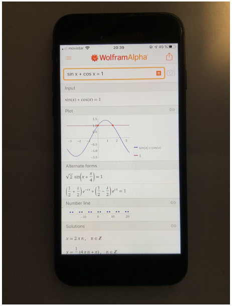
We use as measure of time complexity or cost the function

$$T(n) = \max_{|x|=n} t_{\mathcal{A}}(x)$$

.

Time complexity

The time complexity must be independent of the "used" machine?



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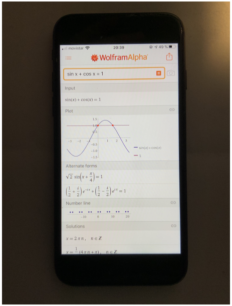
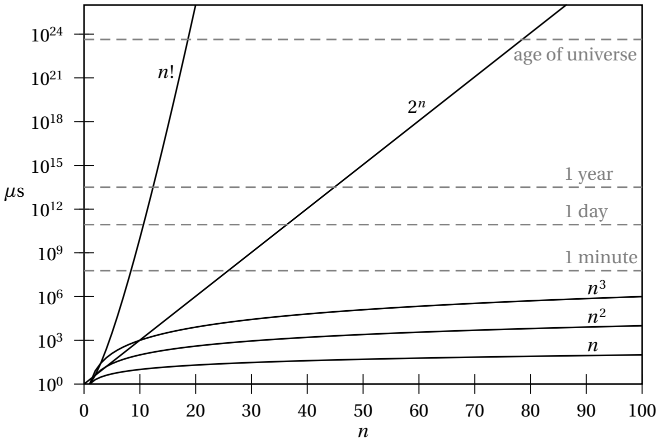


Must we consider carefully how operations scale with respect to size?

Computation time assuming that an input with size $n = 1$ can be solved in 1 μsecond:



From: Moore-Mertens, The Nature of Computation

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Such algorithms are classified as pseudopolynomial.

Efficient algorithms and practical algorithms

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- In this course, we will not enter in the analysis up to constants, but keep in mind that constants matter!!!!
- In practice, even a feasible algorithms with time complexity of for example n^4 could be too slow for $n \geq 1000$.

Asymptotic notation

Symbol	$L = \lim_{n\to\infty} \frac{f(n)}{g(n)}$	intuition
$f(n) = O(g(n))$	$L < \infty$	$f \leq g$
$f(n) = \Omega(g(n))$	$L > 0$	$f \geq g$
$f(n) = \Theta(g(n))$	$0 < L < \infty$	$f = g$
$f(n) = o(g(n))$	$L = 0$	$f < g$
$f(n) = \omega(g(n))$	$L = \infty$	$f > g$

Names used for specific function classes

Name	Definition
polylogarithmic	$f = O(\log^c n)$ (c constant)
polynomial	$f = O(n^c)$ (c constant) or $n^{O(1)}$
subexponential	$f = o(2^{n^\epsilon})$ $(0 < \epsilon < 1)$
exponential	$f = 2^{\mathrm{poly}(n)}$
double exponential	$f = 2^{\exp(n)}$

Notation:

$\lg \equiv \log_2$; $\ln \equiv \log_e$; $\log \equiv \log_{10}$.

Some math you should remember

Given an integer $n > 0$ and a real $a > 1$ and $a \neq 0$:

- Arithmetic summation: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.
- Geometric summation: $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$.

Logarithms and Exponents: For $a, b, c \in \mathbb{R}^+$,

- $\log_b a = c \Leftrightarrow a = b^c \Rightarrow \log_b 1 = 0$
- $\log_b ac = \log_b a + \log_b c$, $\log_b a/c = \log_b a - \log_b c$.
- $\log_b a^c = c \log_b a \Rightarrow c^{\log_b a} = a^{\log_b c} \Rightarrow 2^{\log_2 n} = n$.
- $\log_b a = \log_c a / \log_c b \Rightarrow \log_b a = \Theta(\log_c a)$

Stirling: $n! = \sqrt{2\pi n}(n/e)^n + 0(1/n) + \gamma \Rightarrow n! + \omega((n/2)^n)$.

n-Harmonic: $H_n = \sum_{i=1}^{n} 1/i \sim \ln n$.

Graphs

See for ex. Chapter 3 of Dasgupta, Papadimitriou, Vazirani (DPV).

Graph: $G = (V, E)$, where V is the set of vertices, $n = |V|$, and $E \subset V \times V$ is the set of edges, $m = |E|$,

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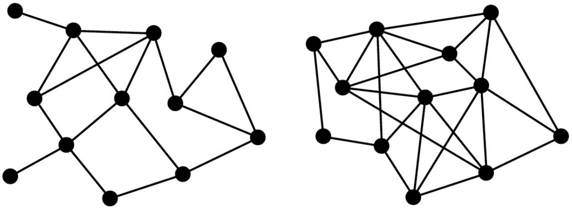
- Graphs: *undirected graphs (graphs)* and *directed graphs (digraphs)*
- The degree of v ($d(v)$) is the number of edges which are incident to v.
- A clique on n vertices (K_n) is a complete graph (with $m = n(n-1)/2$).
- A undirected G is said to be connected if there is a path between any two distinct vertices.
- If G is connected, then $m \geq n - 1$.

Directed graphs

- Edges are directed.
- The connectivity concept in digraphs is the so called strong connectivity: There is is a directed path between any two vertices.
- In a digraph $m \leq n(n-1)$.

Density of a graph

A G with n vertices is said to be dense when $m = \Theta(n^2)$.
When $m = o(n^2)$, G is said to be sparse.



Common graph's data structures

Let $G = (V, E)$ be a graph.

Adjacency list uses $O(|V| + |E|)$ space.

Adjacency matrix uses $O(|V|^2)$ space.

Adjacency matrix

- If G is undirected, its adjacency matrix $A(G)$ is symmetric.
- If A is the adjacency matrix of G, then A^2 gives, for $i, j \in V$, whether there is a path between i and j in G, with length 2.
  For $k > 0$, A^k indicates if there is a path with length k from i to j.
- If G has weights on edges, i.e. $w_{i,j}$ for each $(i, j) \in E$, $A(G)$ keeps w_{ij} in position (i, j).
- Adjacency matrices allow the use of tools from linear algebra.

Complexity issues between matrix and list DS

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Basic graph algorithms

1. Breadth First Search
2. Depth First Search
3. Topological sort in directed acyclic graphs
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5. Strongly connected components in directed graphs

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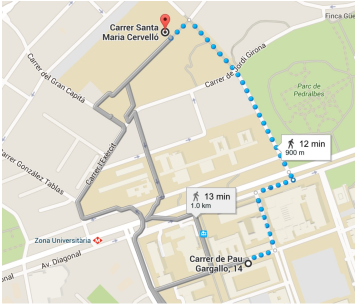
All of them have cost $O(|V| + |E|)$ on graphs given by adjacency lists.

Searching a graph: Breadth First Search

1. Start with vertex v, visit v and all its neighbors.
2. Then, the non-visited neighbors of visited ones.
3. Repeat until all vertices are visited.



BFS use a QUEUE, (FIFO) to keep the neighbors of visited vertices.

Recall that vertices are labeled to avoid visiting them more than once.

Searching a graph: Depth First Search

explore

1. From current vertex, move to a neighbor.
2. Until you get stuck.
3. Then backtrack till new place to explore.

DFS use a STACK, (LIFO)

Time Complexity of DFS and BFS

For graphs given by adjacency lists: $O(|V| + |E|)$

For graphs given by adjacency matrix: $O(|V|^2)$

Therefore, both procedures can be implemented in linear time with respect to the size of the input graph.

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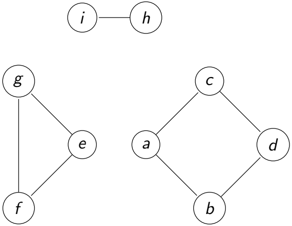
To find connected components in G use DFS/BFS and keep track of the set of vertices visited in each **explore** call.



The problem can be solved in $O(|V| + |E|)$.

Strongly connected components in a digraph

A digraph $G = (V, E)$, is *strongly connected*, if for all $u, v \in V$, there are paths $u \to v$ and $v \to u$.

A strongly connected component is a maximal strongly connected graph.

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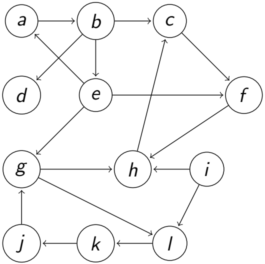
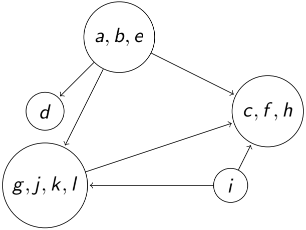
Tarjan's algorithm: Using DFS (once). Complexity $T(n) = O(|V| + |E|)$

Strongly connected components in a digraph

A nice property: For every digraph, the graph on its strongly connected components is *acyclic*.





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- It is easy to se that P$\subseteq$ NP, but it is an open problem to prove that P$=$NP or that P$\neq$NP.

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- Recall that a problem belong to the class P if there exists an algorithm that is polynomial in the worst-case analysis, (for the worst input given by a malicious adversary)
- A problem given in decisional form belong to the class NP non-deterministic polynomial time if, given a certificate of a solution, we can verify in polynomial time that indeed the certificate is a valid solution to the problem and those certificates have polynomial size
- It is easy to se that P$\subseteq$ NP, but it is an open problem to prove that P$=$NP or that P$\neq$NP.
- The class NP-complete is the class of most difficult problems in decisional form that are in NP. Most difficult in the sense that if one of them is proved to be in P then P$=$NP.

Beyond worst-case analysis

- Under the hypothesis that P$\neq$NP, if the decision version of a problem is NP-complete, then the optimization problem will require at least exponential time, for some inputs.
- The classification of a problem as NP-complete is a case of worst-case analysis, and for many problems the "expensive inputs" are few, and far from practical typical inputs. We will see some examples through the course.
- Therefore, there are alternative ways to get in practice, solutions for NP-complete problems, with the use of alternative algorithmic techniques, as approximation (we will see some examples), heuristics and self-learning algorithms, that are deferred to other courses.

A powerful tool to solve problems: Reductions

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A polynomial time reduction $A \leq_m B$ is a function f, computable in polynomial tima such that f maps any input x to A, to an input $f(x)$ to B in such a way that x is a yes input for A iff $f(x)$ is a yes input for B.

We have to extend the concept to function problems.

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Given problems A and B, assume we have an algorithm $\mathcal{A}_B$ to solve the problem B on any input y.

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Therefore if we have that $A \leq B$, if there is an algorithm to solve problem B in polynomial time, then we have an algorithm to solve A in polynomial time.

Evaluating a Boolean circuit

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